

Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2004
MRI Lecture 2

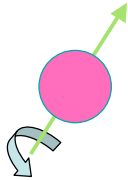
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Today's Topics

- Bloch Equation
- Gradients
- Signal Equation
- k-space trajectories
- Spin-echo/ gradient echo

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Magnetic Moment and Angular Momentum



A charged sphere spinning about its axis has angular momentum and a magnetic moment.

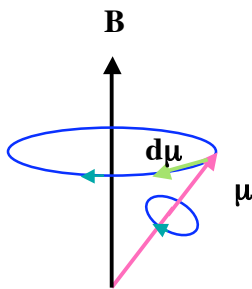
This is a classical analogy that is useful for understanding quantum spin, but remember that it is only an analogy!

Relation: $\boldsymbol{\mu} = \gamma \mathbf{S}$ where γ is the gyromagnetic ratio and \mathbf{S} is the spin angular momentum.

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Precession

$$\frac{d\boldsymbol{\mu}}{dt} = \boldsymbol{\mu} \times \gamma \mathbf{B}$$



Analogous to motion of a gyroscope

Precesses at an angular frequency of

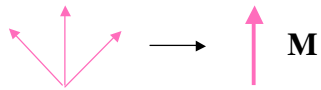
$$\omega = \gamma B$$

This is known as the **Larmor** frequency.

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Magnetization Vector

$$\mathbf{M} = \frac{1}{V} \sum_{\text{protons in } V} \mu_i$$



Vector sum of the magnetic moments over a volume.

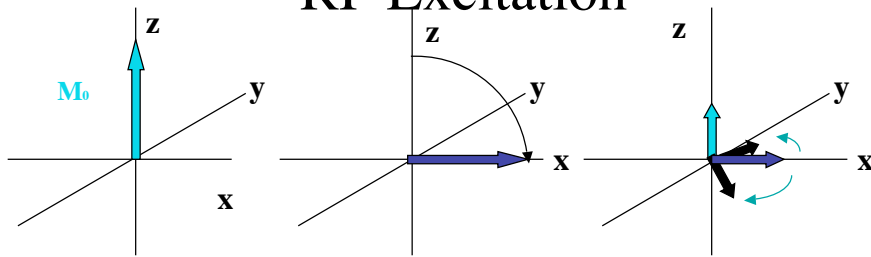
For a sample at equilibrium in a magnetic field, the transverse components of the moments cancel out, so that there is only a longitudinal component.

$$\frac{d\mathbf{M}}{dt} = \gamma \mathbf{M} \times \mathbf{B}$$

Equation of motion is the same form as for individual moments.

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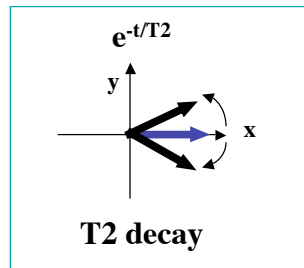
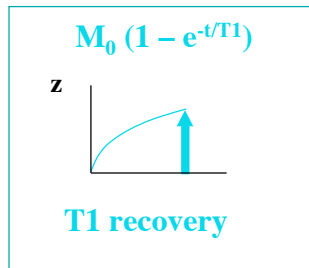
RF Excitation



Doing nothing

Excitation

Relaxation



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Credit: Larry Frank

Bloch Equation

$$\frac{d\mathbf{M}}{dt} = \underbrace{\mathbf{M} \times \gamma \mathbf{B}}_{\text{Precession}} - \underbrace{\frac{M_x \mathbf{i} + M_y \mathbf{j}}{T_2}}_{\text{Transverse Relaxation}} - \underbrace{\frac{(M_z - M_0) \mathbf{k}}{T_1}}_{\text{Longitudinal Relaxation}}$$

$\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors in the x,y,z directions.

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Free precession about static field

$$\begin{aligned} \frac{d\mathbf{M}}{dt} &= \mathbf{M} \times \gamma \mathbf{B} \\ &= \gamma \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ M_x & M_y & M_z \\ B_x & B_y & B_z \end{vmatrix} \\ &= \gamma \begin{pmatrix} \hat{i}(B_z M_y - B_y M_z) \\ -\hat{j}(B_z M_x - B_x M_z) \\ \hat{k}(B_y M_x - B_x M_y) \end{pmatrix} \end{aligned}$$

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Free precession about static field

$$\begin{aligned} \begin{bmatrix} dM_x/dt \\ dM_y/dt \\ dM_z/dt \end{bmatrix} &= \gamma \begin{bmatrix} B_z M_y - B_y M_z \\ B_x M_z - B_z M_x \\ B_y M_x - B_x M_y \end{bmatrix} \\ &= \gamma \begin{bmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} \end{aligned}$$

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Precession

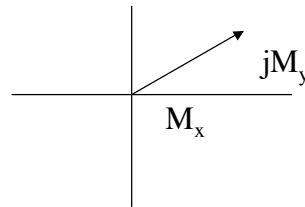
$$\begin{bmatrix} dM_x/dt \\ dM_y/dt \\ dM_z/dt \end{bmatrix} = \gamma \begin{bmatrix} 0 & B_0 & 0 \\ -B_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$

Useful to define $M \equiv M_x + jM_y$

$$\begin{aligned} dM/dt &= d/dt(M_x + jM_y) \\ &= -j\gamma B_0 M \end{aligned}$$

Solution is a time-varying phasor

$$M(t) = M(0)e^{-j\gamma B_0 t} = M(0)e^{-j\omega_0 t}$$



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Precession

$$M(t) = M(0)e^{-j\omega_0 t}$$

$$= (M_x(0)\cos\omega_0 t + M_y(0)\sin\omega_0 t) + j(M_y(0)\cos\omega_0 t - M_x(0)\sin\omega_0 t)$$

In matrix form this is
$$\begin{bmatrix} M_x(t) \\ M_y(t) \end{bmatrix} = \begin{bmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\sin\omega_0 t & \cos\omega_0 t \end{bmatrix} \begin{bmatrix} M_x(0) \\ M_y(0) \end{bmatrix}$$

The full solution is then a rotation about the z-axis.

$$\begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} = \begin{bmatrix} \cos\omega_0 t & \sin\omega_0 t & 0 \\ -\sin\omega_0 t & \cos\omega_0 t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_x(0) \\ M_y(0) \\ M_z(0) \end{bmatrix}$$

$$= R_z(\omega_0 t) \begin{bmatrix} M_x(0) \\ M_y(0) \\ M_z(0) \end{bmatrix}$$

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Matrix Form with $B=B_0$

$$\begin{bmatrix} dM_x/dt \\ dM_y/dt \\ dM_z/dt \end{bmatrix} = \begin{bmatrix} -1/T_2 & \gamma B_0 & 0 \\ -\gamma B_0 & 1/T_2 & 0 \\ 0 & 0 & -1/T_1 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_0/T_1 \end{bmatrix}$$

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Z-component solution

$$M_z(t) = M_0 + (M_z(0) - M_0)e^{-t/T_1}$$

Saturation Recovery

$$\text{If } M_z(0) = 0 \text{ then } M_z(t) = M_0(1 - e^{-t/T_1})$$

Inversion Recovery

$$\text{If } M_z(0) = -M_0 \text{ then } M_z(t) = M_0(1 - 2e^{-t/T_1})$$

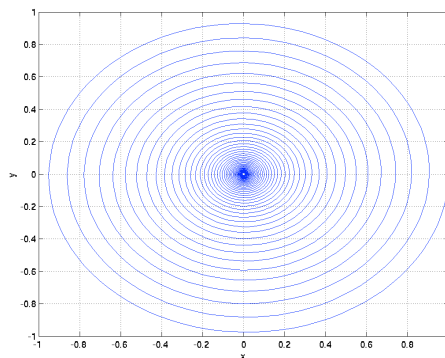
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Transverse Component

$$M \equiv M_x + jM_y$$

$$\begin{aligned} dM/dt &= d/dt(M_x + iM_y) \\ &= -j(\omega_0 + 1/T_2)M \end{aligned}$$

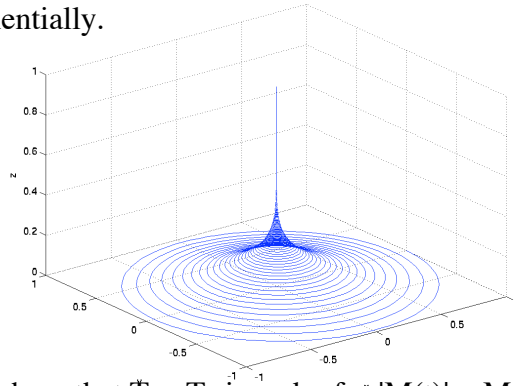
$$M(t) = M(0)e^{-j\omega_0 t} e^{-t/T_2}$$



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Summary

- 1) Longitudinal component recovers exponentially.
- 2) Transverse component precesses and decays exponentially.



Fact: Can show that $T_2 < T_1$ in order for $|M(t)| \leq M_0$
Physically, the mechanisms that give rise to T_1 relaxation also contribute to transverse T_2 relaxation.

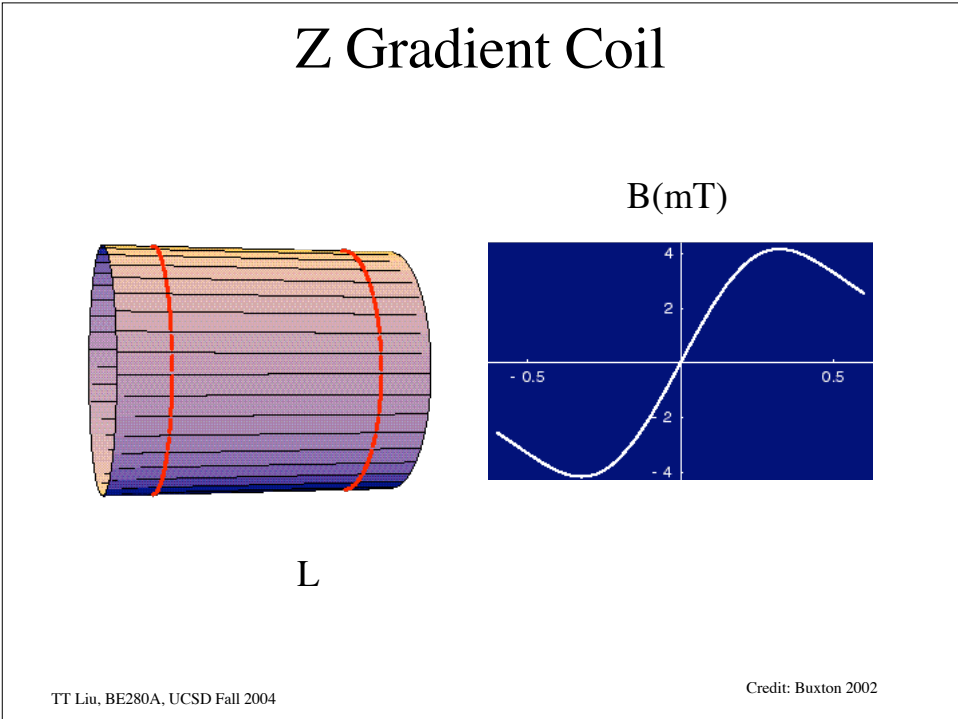
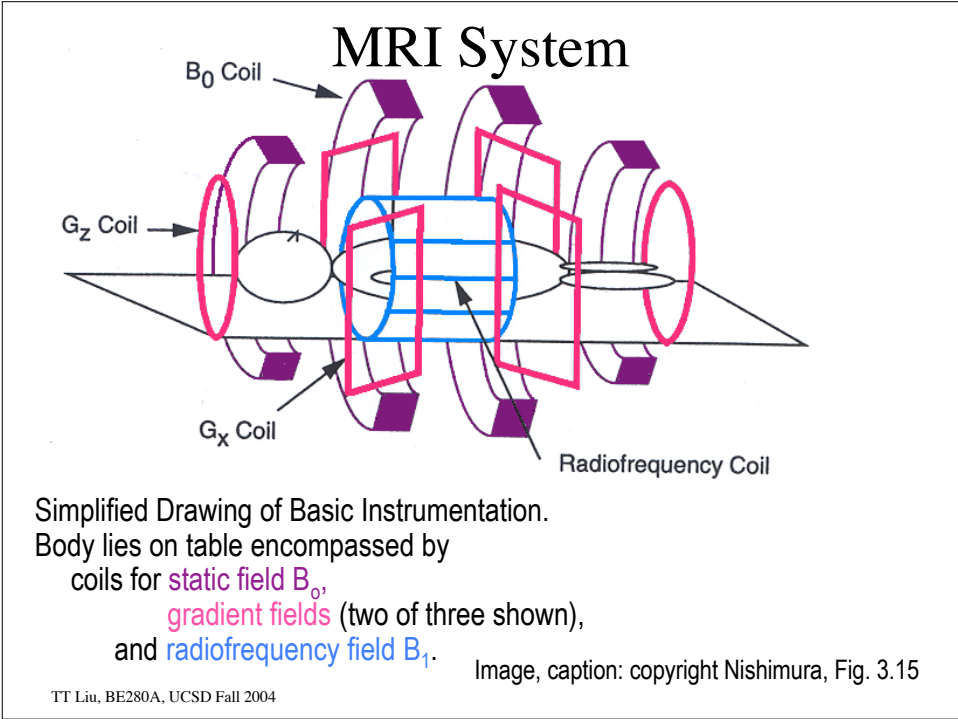
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Gradients

Spins precess at the Larmor frequency, which is proportional to the local magnetic field. In a constant magnetic field $B_z = B_0$, all the spins precess at the same frequency (ignoring chemical shift).

Gradient coils are used to add a spatial variation to B_z such that $B_z(x, y, z) = B_0 + \Delta B_z(x, y, z)$. Thus, spins at different physical locations will precess at different frequencies.

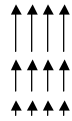
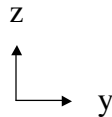
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Gradient Fields

$$B_z(x, y, z) = B_0 + \frac{\partial B_z}{\partial x} x + \frac{\partial B_z}{\partial y} y + \frac{\partial B_z}{\partial z} z$$

$$= B_0 + G_x x + G_y y + G_z z$$



$$G_z = \frac{\partial B_z}{\partial z} > 0$$



$$G_y = \frac{\partial B_z}{\partial y} > 0$$

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Gradient Fields

Define

$$\vec{G} \equiv G_x \hat{i} + G_y \hat{j} + G_z \hat{k} \quad \vec{r} \equiv x \hat{i} + y \hat{j} + z \hat{k}$$

So that

$$G_x x + G_y y + G_z z = \vec{G} \cdot \vec{r}$$

Also, let the gradient fields be a function of time. Then the z-directed magnetic field at each point in the volume is given by :

$$B_z(\vec{r}, t) = B_0 + \vec{G}(t) \cdot \vec{r}$$

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Static Gradient Fields

In a uniform magnetic field, the transverse magnetization is given by:

$$M(t) = M(0)e^{-j\omega_0 t} e^{-t/T_2}$$

In the presence of non time-varying gradients we have

$$\begin{aligned} M(\vec{r}) &= M(\vec{r}, 0)e^{-j\gamma B_z(\vec{r})t} e^{-t/T_2(\vec{r})} \\ &= M(\vec{r}, 0)e^{-j\gamma(B_0 + \vec{G} \cdot \vec{r})t} e^{-t/T_2(\vec{r})} \\ &= M(\vec{r}, 0)e^{-j\omega_0 t} e^{-j\gamma \vec{G} \cdot \vec{r} t} e^{-t/T_2(\vec{r})} \end{aligned}$$

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Time-Varying Gradient Fields

In the presence of time-varying gradients the frequency as a function of space and time is:

$$\begin{aligned} \omega(\vec{r}, t) &= \gamma B_z(\vec{r}, t) \\ &= \gamma B_0 + \gamma \vec{G}(t) \cdot \vec{r} \\ &= \omega_0 + \Delta\omega(\vec{r}, t) \end{aligned}$$

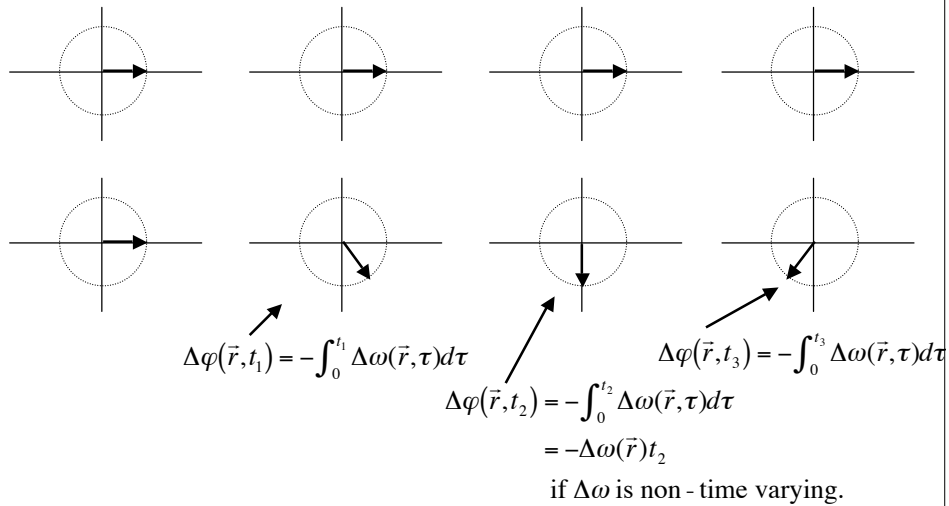
The phase of each spin is $\varphi(\vec{r}, t) = -\int_0^t \omega(\vec{r}, \tau) d\tau$
 $= -\omega_0 t + \Delta\varphi(\vec{r}, t)$

Where the incremental phase due to the gradients is

$$\begin{aligned} \Delta\varphi(\vec{r}, t) &= -\int_0^t \Delta\omega(\vec{r}, \tau) d\tau \\ &= -\int_0^t \gamma \vec{G}(\vec{r}, \tau) \cdot \vec{r} d\tau \end{aligned}$$

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Time-Varying Gradient Fields



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Time-Varying Gradient Fields

The transverse magnetization is then given by

$$\begin{aligned}
 M(\vec{r}, t) &= M(\vec{r}, 0)e^{-t/T_2(\vec{r})} e^{i\varphi(\vec{r}, t)} \\
 &= M(\vec{r}, 0)e^{-t/T_2(\vec{r})} e^{-j\omega_0 t} \exp\left(-j \int_0^t \Delta\omega(\vec{r}, \tau) d\tau\right) \\
 &= M(\vec{r}, 0)e^{-t/T_2(\vec{r})} e^{-j\omega_0 t} \exp\left(-j\gamma \int_0^t \vec{G}(\tau) \cdot \vec{r} d\tau\right)
 \end{aligned}$$

It can be shown that this satisfies the differential equation

$$\begin{aligned}
 dM/dt &= d/dt(M_x + iM_y) \\
 &= -j(\omega_0 + 1/T_2)M
 \end{aligned}$$

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Signal Equation

Signal from a volume

$$s_r(t) = \int_V M(\vec{r}, t) dV$$

$$= \int_x \int_y \int_z M(x, y, z, 0) e^{-t/T_2(\vec{r})} e^{-j\omega_0 t} \exp\left(-j\gamma \int_0^t \vec{G}(\tau) \cdot \vec{r} d\tau\right) dx dy dz$$

For now, consider signal from a slice along z and drop the T_2 term. Define

$$m(x, y) \equiv \int_{z_0 - \Delta z/2}^{z_0 + \Delta z/2} M(\vec{r}, t) dz$$

To obtain

$$s_r(t) = \int_x \int_y m(x, y) e^{-j\omega_0 t} \exp\left(-j\gamma \int_0^t \vec{G}(\tau) \cdot \vec{r} d\tau\right) dx dy$$

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Signal Equation

Demodulate the signal to obtain

$$s(t) = e^{j\omega_0 t} s_r(t)$$

$$= \int_x \int_y m(x, y) \exp\left(-j\gamma \int_0^t \vec{G}(\tau) \cdot \vec{r} d\tau\right) dx dy$$

$$= \int_x \int_y m(x, y) \exp\left(-j\gamma \int_0^t [G_x(\tau)x + G_y(\tau)y] d\tau\right) dx dy$$

$$= \int_x \int_y m(x, y) \exp\left(-j2\pi(k_x(t)x + k_y(t)y)\right) dx dy$$

Where

$$k_x(t) = \frac{\gamma}{2\pi} \int_0^t G_x(\tau) d\tau$$

$$k_y(t) = \frac{\gamma}{2\pi} \int_0^t G_y(\tau) d\tau$$

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MR signal is Fourier Transform

$$\begin{aligned} s(t) &= \int_x \int_y m(x, y) \exp(-j2\pi(k_x(t)x + k_y(t)y)) dx dy \\ &= M(k_x(t), k_y(t)) \\ &= F[m(x, y)]_{k_x(t), k_y(t)} \end{aligned}$$

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K-space

At each point in time, the received signal is the Fourier transform of the object

$$s(t) = M(k_x(t), k_y(t)) = F[m(x, y)]_{k_x(t), k_y(t)}$$

evaluated at the spatial frequencies:

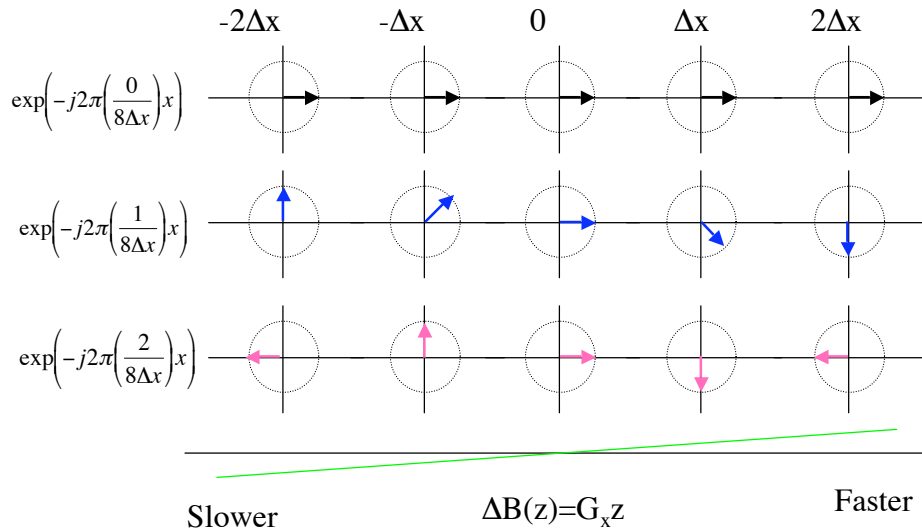
$$k_x(t) = \frac{\gamma}{2\pi} \int_0^t G_x(\tau) d\tau$$

$$k_y(t) = \frac{\gamma}{2\pi} \int_0^t G_y(\tau) d\tau$$

Thus, the gradients control our position in k-space. The design of an MRI pulse sequence requires us to efficiently cover enough of k-space to form our image.

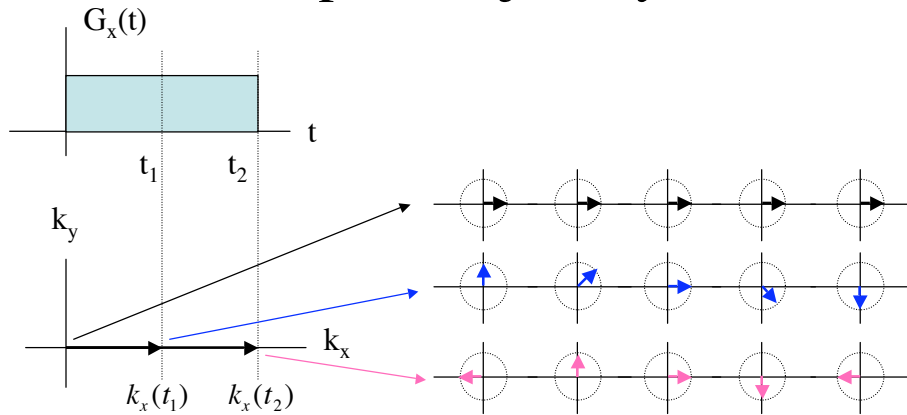
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Interpretation



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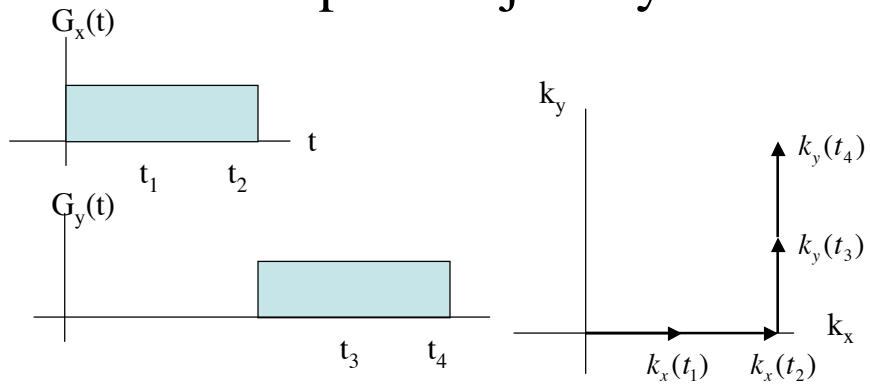
K-space trajectory



$$k_x(t) = \frac{\gamma}{2\pi} \int_0^t G_x(\tau) d\tau$$

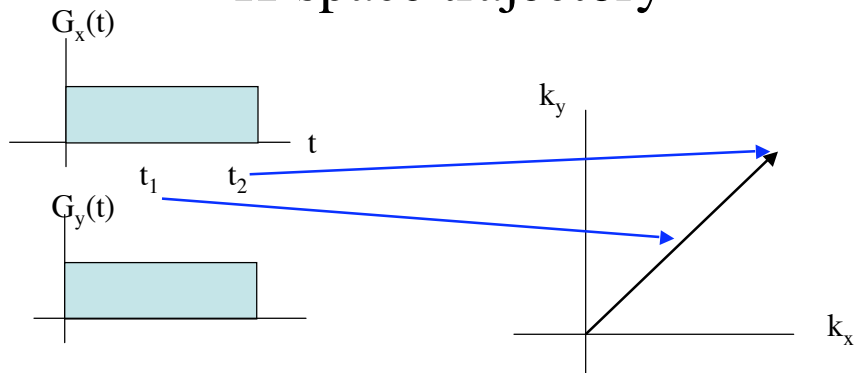
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K-space trajectory



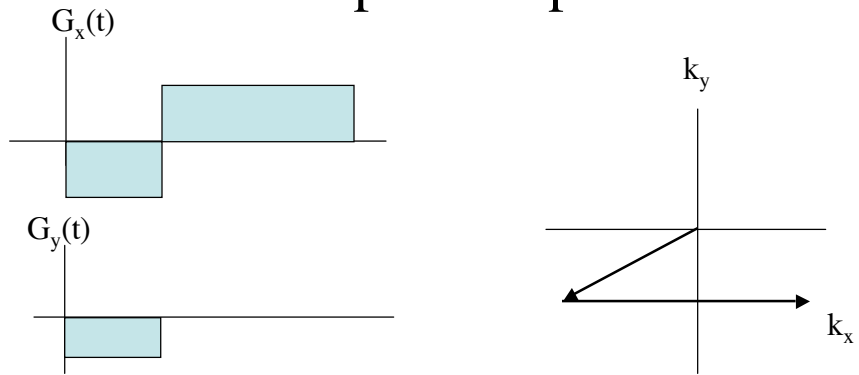
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K-space trajectory



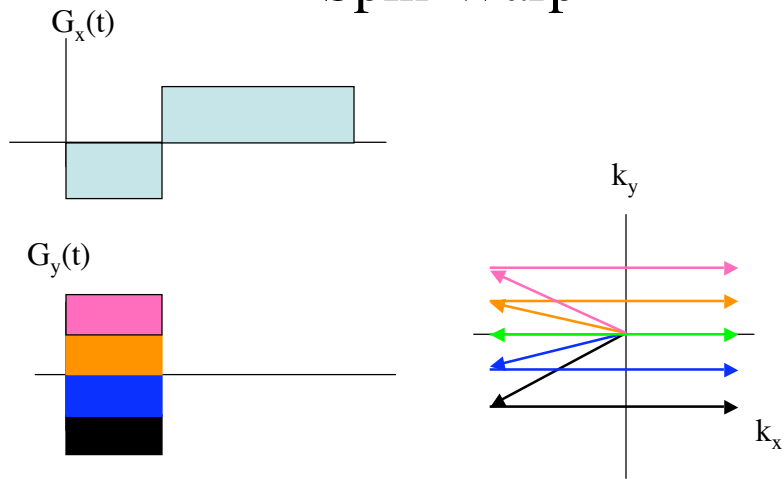
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Spin-Warp



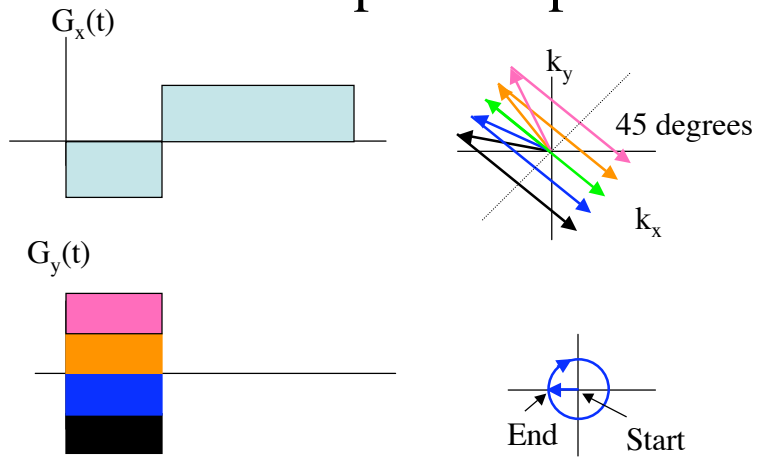
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Spin-Warp



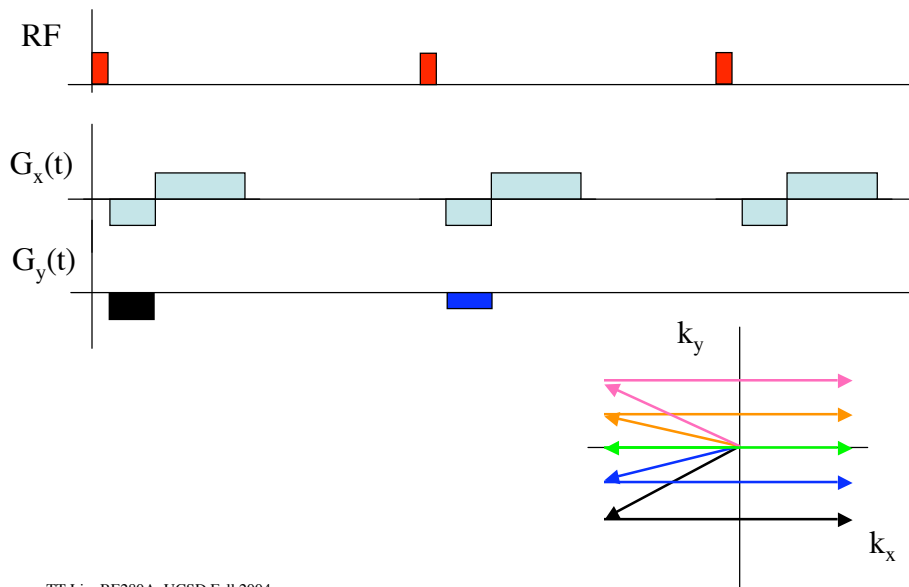
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Spin-Warp



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Spin-Warp Pulse Sequence



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