Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2004
Noise

What is Noise?

Random fluctuations in either the imaging system or the object being imaged.

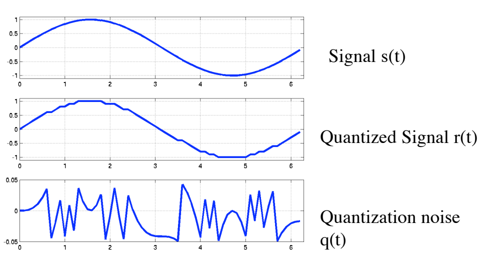
Quantization Noise: Due to conversion from analog waveform to digital number.

Quantum Noise: Random fluctuation in the number of photons emittted and recorded.

Thermal Noise: Random fluctuations present in all electronic systems. Also, sample noise in MRI

Other types: flicker, burst, avalanche - observed in semiconductor devices.

Quantization Noise



$r(t) = s(t) + q(t)$
Although the noise is deterministic, it is useful to model the noise as a random process.

Poisson Process

Events occur at random instants of time at an average rate of λ events per second.

Examples: arrival of customers to an ATM, emission of photons from an x-ray source, lightning strikes in a thunderstorm.

Assumptions:

1) Probability of more than 1 event in an small time interval is small.
2) Probability of event occurring in a given small time interval is independent of another event occuring in other small time intervals.

Poisson Process

$$P\left[N(t)=k\right]=\frac{(\lambda t)^k}{k!}\exp(-\lambda t)$$

$\lambda =$ Average rate of events per second
λt = Average number of events at time t
λt = Variance in number of events

Probability of interarrival times
$P\left[T>t\right]=e^{-\lambda t}$

Example

A service center receives an average of 15 inquiries per minute. Find the probability that 3 inquiries arrive in the first 10 seconds.

$\lambda = 15/60 = 0.25$
$\lambda t = 0.25(10) = 2.5$

$$P[N(t=10)=3) = \frac{(2.5)^3}{3!}\exp(-2.5) = .2138$$

Quantum Noise

Fluctuation in the number of photons emitted by the x-ray source and recorded by the detector.

$$P_k = \frac{N_0^k \exp(-N_0)}{k!}$$

P_k : Probability of emitting k photons in a given time interval.

N_0 : Average number of photons emitted in that time interval = λt

Transmitted Photons

$$Q_k = \frac{(pN_0)^k \exp(-pN_0)}{k!}$$

Q_k : Probability of k photons making it through object

N_0 : Average number of photons emitted in that time interval = λt

$p = \exp(-\int \mu dz) =$ probability of proton being transmitted

Example

Over the diagnostic energy range, the photon density is approximately 2.5×10^{10} photons/cm^2 / R where R stands for roentgen (unit for X-ray exposure).

A typical chest x-ray has an exposure of 50 mR.
For transmission in regions devoid of bone,
$p = \exp(-\int \mu dz) \approx 0.05$
What are the mean and standard deviation of the number of photons that make it it to a 1 mm^2 detector?

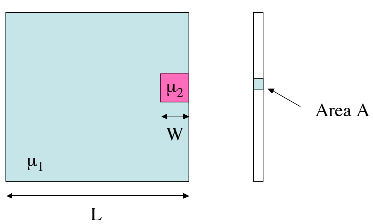
$$pN_0 = 0.05 \cdot 2.5 \times 10^{10} \cdot .050 \cdot (.1)^2 = 6.25 \times 10^5 \text{ photons}$$

mean = 6.25×10^5 photons
standard deviation = $\sqrt{6.25 \times 10^5} = 790$ photons

Contrast and SNR for X-Rays

$$Contrast = C = \frac{\Delta I}{\bar{I}}$$

$$\begin{aligned} SNR &= \frac{\Delta I}{\sigma_I} \\ &= \frac{\text{Mean difference in \# of photons}}{\text{Standard Deviation of \# photons}} \\ &= \frac{CpN_0}{\sqrt{pN_0}} \\ &= C\sqrt{pN_0} \end{aligned}$$



$$C = \frac{\Delta I}{\bar{I}} = \frac{N_0\left(\exp(-\mu_1 L) - \exp\left(-\left(\mu_1(L-W) + \mu_2 W\right)\right)\right)}{N_0 \exp(-\mu_1 L)}$$

$$SNR = \frac{CN_0 A \exp(-\mu_1 L)}{\sqrt{N_0 A \exp(-\mu_1 L)}} = C\sqrt{N_0 A \exp(-\mu_1 L)}$$

Signal to Noise Ratio for CT

$$\begin{aligned} SNR &= \frac{C\bar{\mu}}{\sigma_\mu} \\ &= \frac{C\bar{\mu}}{\sqrt{\frac{\pi^2}{\bar{n}hM}\frac{2\rho_0^3}{3}}} \\ &\propto KCw\bar{\mu}\sqrt{\bar{n}AM} \end{aligned}$$

C = contrast
w = width of detector
$\bar{\mu}$ = mean attenuation
$\bar{n}$ = mean density of transmitted photons
A = area of detector
M = number of views
$\rho_0 = K/w$ = bandwidth of Ram-Lak filter
K = scaling constant, order unity

Thermal Noise

Fluctuations in voltage across a resistor due to random thermal motion of electrons.

Described by J.B. Johnson in 1927 (therefore sometimes called Johnson noise). Explained by H. Nyquist in 1928.

$$\left\langle V^2 \right\rangle = 4kT \cdot R \cdot BW$$

Bandwidth

Variance in Voltage

Resistance

Temperature

Thermal Noise

$\left\langle V^2 \right\rangle = 4kT \cdot R \cdot BW$

At room temperature, noise in a 1 kΩ resistor is
$\left\langle V^2 \right\rangle / BW = 16 \times 10^{-18} \ V^2 / Hz$

In root mean squared form, this corresponds to
$V/BW = 4 \ nV/\sqrt{Hz}$.

Example: For BW = 250 kHz and 2 kΩ resistor, total noise voltage is
$\sqrt{2 \cdot 16 \times 10^{-18} \cdot 250 \times 10^{3}} = 4 \ \mu V$

Thermal Noise

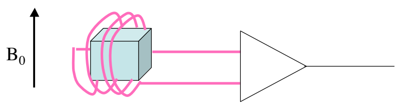
Noise spectral density is independent of frequency up to 10^{13} Hz. Therefore it is a source of white noise.

Amplitude distribution of the noise is Gaussian.

Signal in MRI

Recall the signal equation has the form

$$s_r(t) = \int\int\int M(x,y,z) e^{-t/T_2(\mathbf{r})} e^{-j\omega_0 t} \exp\left(-j\gamma \int_0^t \mathbf{G}(\tau)\cdot \mathbf{r}(\tau)\, d\tau\right) dx\,dy\,dz$$



Faraday's Law

$$EMF = -\frac{\partial \phi}{\partial t}$$

$$\phi = \text{Magnetic Flux} = \int \mathbf{B}_1(x,y,z)\cdot \mathbf{M}(x,y,z)\, dV$$

Signal in MRI

Signal in the receiver coil

$$s_r(t) = j\omega_0 B_{1xy} \int M(x,y,z) e^{-t/T_2(\mathbf{r})} e^{-j\omega_0 t} \exp\left(-j\gamma \int_0^t \mathbf{G}(\tau)\cdot \mathbf{r}(\tau)\, d\tau\right) dV$$

Recall, total magnetization is proportional to B_0

Also $\omega_0 = \gamma B_0$.

Therefore, total signal is proportional to B_0^2

Noise in MRI

Primary sources of noise are:
1) Thermal noise of the receiver coil
2) Thermal noise of the sample.

Coil Resistance: At higher frequencies, the EM waves tend to travel along the surface of the conductor (skin effect). As a result,

$$R_{\text{coil}} \propto \omega_0^{1/2} \Rightarrow \left\langle N_{coil}^2 \right\rangle \propto \omega_0^{1/2} \propto B_0^{1/2}$$

Sample Noise: Noise is white, but differentiation process due to Faraday's law introduces a multiplication by ω_0. As a result, the noise variance from the sample is proportional to ω_0^2.

$$\left\langle N_{sample}^2 \right\rangle \propto \omega_0^2 \propto B_0^2$$

SNR in MRI

$$SNR \propto \frac{B_0^2}{\sqrt{\alpha B_0^{1/2} + \beta B_0^2}}$$

If coil noise dominates

$$SNR \propto B_0^{7/4}$$

If sample noise dominates

$$SNR \propto B_0$$

Random Variables

A random variable X is characterized by its cumulative distribution function (CDF)

$$\Pr(X \leq x) = F_X(x)$$

The derivative of the CDF is the probability density function(pdf)

$$f_X(x) = dF_X(x)/dx$$

The probability that X will take on values between two limits x_1 and x_2 is

$$\Pr(x_1 \leq X \leq x_2) = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)dx$$

Random Processes

A random process is an indexed family of random variables

Examples:
discrete: $X_1, X_2, \ldots, X_N$
continuous: $X(t)$

If all the random variables share the same pdf and take on values independently, the process is said to be independent and identically distributed (iid).

Example: unbiased coin tosses

If the joint statistics of the process do not vary with index, the process is said to be stationary.

Random Processes

Correlation

$R(t_1,t_2) = E(X(t_1)X^*(t_2))$

$R(i,j) = E(X_i X_j)$

Covariance

$C(t_1,t_2) = E\left(\left(X(t_1) - \overline{X}(t_1)\right)\left(X(t_2) - \overline{X}(t_2)\right)^*\right)$

$C(i,j) = E\left(\left(X_i - \overline{X}_i\right)\left(X_j - \overline{X}_j\right)\right)$

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Power Spectral Density

For stationary process

$R(t,t_2) = R(\tau) = E(X(t)X^*(t+\tau))$ for $\tau = t_2 - t$

$R(i,j) = R(m) = E(X_i X_{i+m})$ for $m = j - i$

Power Spectral Density

$S_X(f) = F\{R(\tau)\}$

$S_X(f) = F\{R(m)\}$

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Example

Thermal noise has a flat power spectrum over the range of frequencies of interest.
So,

$\mathrm{S}_X(f) = \sigma_0^2$

Therefore

$R_X(\tau) = \sigma_0^2 \delta(\tau)$

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Vector Notation

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}$$

$$\mathbf{R} = E(\mathbf{X}\mathbf{X}^H) = E\left(\begin{bmatrix} X_1X_1^* & X_1X_2^* & \cdots & X_1X_N^* \\ X_2X_1^* & X_2X_2^* & \cdots & X_2X_N^* \\ \vdots & \vdots & \ddots & \vdots \\ X_NX_1^* & X_NX_2^* & \cdots & X_NX_N^* \end{bmatrix}\right)$$

Example

$\mathbf{X}$ denotes a stationary random process with mean zero and correlation $R[m] = \sigma^2\delta[m]$

$$\mathbf{R} = E(\mathbf{X}\mathbf{X}^H) = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}$$

$$= \sigma^2\mathbf{I}$$

Review: Orthonormal basis

A set of vectors $S = \{\mathbf{b}_i\}$ forms an orthonormal basis, if $\langle \mathbf{b}_i, \mathbf{b}_j \rangle = 0$ for $i \neq j$, every basis vector is normalized to have unit length $\|\mathbf{b}_i\| = 1$, and any vector $\mathbf{y}$ in the space can be expressed as a linear combination of the basis vectors, i.e. $\mathbf{y} = \sum_k c_k \mathbf{b}_k$.

Finding Expansion Coefficients

Define the basis matrix as $\mathbf{B} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_N]$.

Then any vector $\mathbf{y} = \mathbf{Bc} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_N]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}$

Multiply both sides of the equation by $\mathbf{B}^{-1}$, to obtain $\mathbf{c} = \mathbf{B}^{-1}\mathbf{y}$.
Because the basis vectors are orthonormal $\mathbf{B}^H\mathbf{B} = \mathbf{I}$, and therefore $\mathbf{B}^{-1} = \mathbf{B}^H$. So, we can also write $\mathbf{c} = \mathbf{B}^H\mathbf{y}$.
By definition, $\mathbf{B}$ is an orthonormal or unitary matrix.

Expansion Coefficients

$$\mathbf{c} = \mathbf{B}^H\mathbf{y} = \begin{bmatrix} \mathbf{b}_1^H \\ \mathbf{b}_2^H \\ \vdots \\ \mathbf{b}_N^H \end{bmatrix}\mathbf{y} = \begin{bmatrix} \langle \mathbf{b}_1, \mathbf{y} \rangle \\ \langle \mathbf{b}_2, \mathbf{y} \rangle \\ \vdots \\ \langle \mathbf{b}_N, \mathbf{y} \rangle \end{bmatrix}$$

For any vector **y,** the *i*th expansion coefficient is the inner product of the *i*th orthonormal basis vector with **y**.

Noise after inverse transform

Let the coefficients be described by a zero-mean, stationary random process $\mathbf{C}$ with correlation matrix $\mathbf{R_C} = \sigma^2\mathbf{I}$

Now let $\mathbf{X} = \mathbf{BC}$, then

$$\begin{aligned} \mathbf{R}_X &= E\left(\mathbf{XX}^H\right) \\ &= E\left(\mathbf{BCC}^H\mathbf{B}^H\right) \\ &= \mathbf{B}E\left(\mathbf{CC}^H\right)\mathbf{B}^H \\ &= \sigma^2\mathbf{BB}^H \\ &= \sigma^2\mathbf{I} \end{aligned}$$

Note: From orthonormality of basis functions $\mathbf{B}^H\mathbf{B} = \mathbf{I}$.
Therefore $\mathbf{BB}^H\mathbf{BB}^H = \mathbf{BB}^H$, so $\mathbf{BB}^H = \mathbf{I}$.

DFT Basis Functions

DFT: $G[m] = \sum_{n=0}^{N-1} g[n] e^{-j2\pi mn/N}$

Basis Functions are therefore:

$b_m[n] = e^{j2\pi mn/N}$

Inverse DFT: $g[n] = \frac{1}{N}\sum_{m=0}^{N-1} G[m] e^{j2\pi mn/N}$

Noise after inverse transform

Let the coefficients be described by a zero-mean, stationary random process $\mathbf{C}$ with correlation matrix $\mathbf{R}_\mathbf{C} = \sigma^2\mathbf{I}$

Now let $\mathbf{X} = \frac{1}{N}\mathbf{BC}$, then

$$\begin{aligned}\mathbf{R}_X &= N^{-2}E\left(\mathbf{XX}^H\right)\\ &= N^{-2}E\left(\mathbf{BCC}^H\mathbf{B}^H\right)\\ &= N^{-2}\mathbf{B}E\left(\mathbf{CC}^H\right)\mathbf{B}^H\\ &= \sigma^2 N^{-2}\mathbf{BB}^H\\ &= \frac{\sigma^2}{N}\mathbf{I}\end{aligned}$$

Note: $\mathbf{BB}^H = N\mathbf{I}$.

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Noise in k-space

Recall that in MRI we acquire samples in k-space. The noise in these samples is typically well described by an iid random process.
For Cartesian sampling, the noise in the image domain is then also described by an iid random process.

For each point in k-space, $SNR = \frac{S(k)}{\sigma_n}$ where $S(k)$ is the signal and σ_n is the standard deviation of each noise sample.

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Noise in image space

If noise variance per sample in k-space is σ_n^2.

Noise variance per sample in image space σ_n^2 / N.

$$SNR \propto \frac{S_0}{\sigma_n / \sqrt{N}} = \sqrt{N}\,\frac{S_0}{\sigma_n}$$

Signal Averaging

We can improve SNR by acquiring additional k-space measurements.

Consider two measurements of a point in k-space with values

$y_1 = y_0 + n_1$
$y_2 = y_0 + n_2$

The sum of the two measurements is $2y_0 + (n_1 + n_2)$.

If the noise in the measurements is independent, then the variances sum and the total variance is $2\sigma_n^2$

$$SNR_{Tot} = \frac{2y_0}{\sqrt{2}\sigma_n} = \sqrt{2}SNR_{original}$$

In general, $SNR \propto \sqrt{N_{ave}}$

Effect of Readout Window

ADC samples acquired with sampling period Δt.

Thermal noise per sample is $\sigma_n^2 = \Delta f = \dfrac{1}{\Delta \mathrm{t}}$

If we double length of the readout window, the noise variance per sample decreases by two.

The noise standard deviation decreases by $\sqrt{2}$, and the SNR increases by $\sqrt{2}$.

In general, SNR $\propto \sqrt{T_{\mathrm{Re}ad}} = \sqrt{N_{k_x}\Delta t}$

SNR and Phase Encodes

Assume that spatial resolution is held constant. What happens if we increase the number of phase encodes? Recall that $\delta_y = \frac{1}{W_{k_y}}$. Thus, increasing the number of phase encodes N_{PE}, decreases Δk_y and increases FOV_y.

If we double the number of phase encodes, each point in image space has double the number of k-space lines contributing to its signal. The noise variances sum. The SNR therefore goes up by $\sqrt{2}$.

In general $SNR \propto \sqrt{N_{PE}}$

Overall SNR

$$SNR \propto \frac{Signal}{\sigma_n} \propto \frac{\Delta x \Delta y \Delta z}{\sigma_n}$$

Putting everything together, we find that

$$SNR \propto \sqrt{N_{ave} N_x N_{PE} \Delta t}\, \Delta x \Delta y \Delta z$$

$$= \sqrt{Measurement\ Time} \cdot Voxel\ Volume$$

In general,

$$SNR \propto \sqrt{Measurement\ Time} \cdot Voxel\ Volume \cdot f(\rho, T_1, T_2)$$