Bioengineering 280A
Principles of Biomedical Imaging

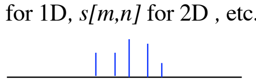
Fall Quarter 2004
Lecture 2
Linear Systems

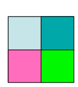
Topics

1. Linearity
2. Impulse Response and Delta functions
3. Superposition Integral
4. Shift Invariance
5. 1D and 2D convolution
6. Examples.

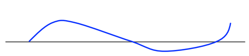
Signals and Images

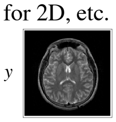
Discrete-time/space signal/image: continuous valued function with a discrete time/space index, denoted as *s[n]* for 1D, *s[m,n]* for 2D , etc.



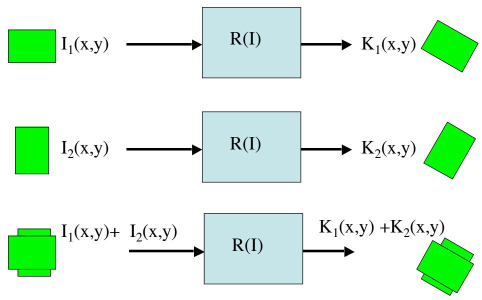


Continuous-time/space signal/image: continuous valued function with a continuous time/space index, denoted as $s(t)$ or $s(x)$ for 1D, $s(x,y)$ for 2D, etc.





Linearity (Addition)



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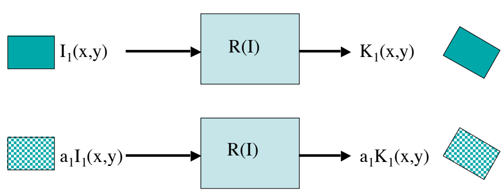
Linearity (Scaling)



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Linearity

A system R is linear if for two inputs $I_1(x,y)$ and $I_2(x,y)$ with outputs

$R(I_1(x,y))=K_1(x,y)$ and $R(I_2(x,y))=K_2(x,y)$

the response to the weighted sum of inputs is the weighted sum of outputs:

$R(a_1I_1(x,y)+ a_2I_2(x,y))=a_1K_1(x,y)+ a_2K_2(x,y)$

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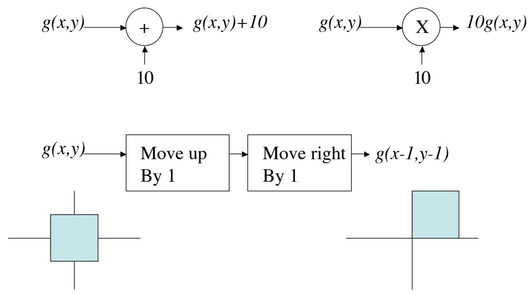
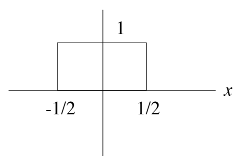
Example

Are these linear systems?



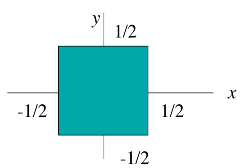
Rectangle Function

$$\Pi(x) = \begin{cases} 0 & |x| > 1/2 \\ 1 & |x| \leq 1/2 \end{cases}$$



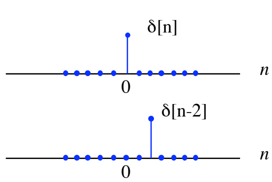
Also called rect(x)

$$\Pi(x,y) = \Pi(x)\Pi(y)$$



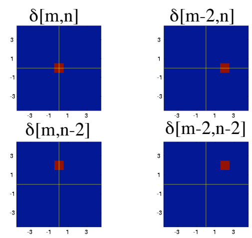
Kronecker Delta Function

$$\delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \textit{otherwise} \end{cases}$$



Kronecker Delta Function

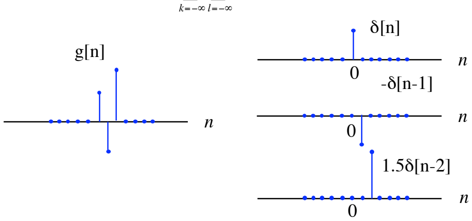
$$\delta[m,n] = \begin{cases} 1 & \text{for } m=0, n=0 \\ 0 & otherwise \end{cases}$$



Discrete Signal Expansion

$$g[n] = \sum_{k=-\infty}^{\infty} g[k]\delta[n-k]$$

$$g[m,n] = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} g[k,l]\delta[m-k,n-l]$$



Dirac Delta Function

Notation:

$\delta(x)$ - 1D Dirac Delta Function

$\delta(x,y)$ or ${}^2\delta(x,y)$ - 2D Dirac Delta Function

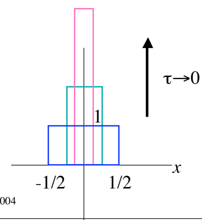
$\delta(x,y,z)$ or ${}^3\delta(x,y,z)$ - 3D Dirac Delta Function

$\delta(\vec{r})$ - N Dimensional Dirac Delta Function

1D Dirac Delta Function

$\delta(x) = 0$ when $x \neq 0$ and $\int_{-\infty}^{\infty} \delta(x)dx = 1$

Can interpret the integral as a limit of the integral of an ordinary function that is shrinking in width and growing in height, while maintaining a constant area. For example, we can use a shrinking rectangle function such that $\int_{-\infty}^{\infty} \delta(x)dx = \lim_{\tau \to 0} \int_{-\infty}^{\infty} \tau^{-1}\Pi(x/\tau)dx$.



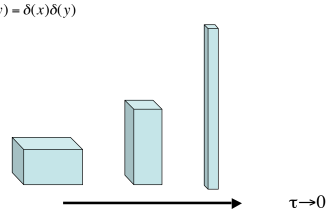
2D Dirac Delta Function

$\delta(x,y) = 0$ when $x^2 + y^2 \neq 0$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)dxdy = 1$

where we can consider the limit of the integral of an ordinary 2D function that is shrinking in width but increasing in height while maintaining constant area.

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)dxdy = \lim_{\tau \to 0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tau^{-2}\Pi(x/\tau, y/\tau)dxdy$.

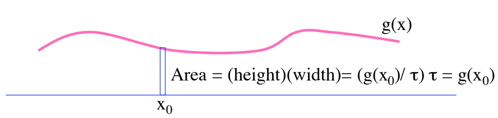
Useful fact : $\delta(x,y) = \delta(x)\delta(y)$



Generalized Functions

Dirac delta functions are not ordinary functions that are defined by their value at each point. Instead, they are generalized functions that are defined by what they do underneath an integral.

The most important property of the Dirac delta is the sifting property $\int_{-\infty}^{\infty} \delta(x - x_0)g(x)dx = g(x_0)$ where $g(x)$ is a smooth function. This sifting property can be understood by considering the limiting case $\lim_{\tau \to 0} \int_{-\infty}^{\infty} \tau^{-1}\Pi(x/\tau)g(x)dx = g(x_0)$



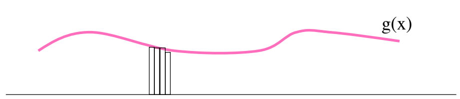
Working with Dirac Delta Functions

What is δ(ax - b)? What is dδ(x)/dx?
How do we define generalized functions?
There are two main approaches:
1) Look at the limit of an integral with sequences.
2) Consider the behavior of the function when integrated with a *nice* test function. Two generalized functions $\delta_1(t)$ and $\delta_2(t)$ are equivalent in the distributional sense when $\int_{-\infty}^{\infty}\delta_1(t)\phi(t)dt = \int_{-\infty}^{\infty}\delta_2(t)\phi(t)dt$

Example: $\delta(ax) = ??$

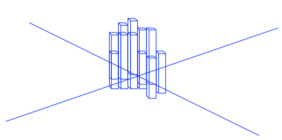
Representation of 1D Function

From the sifting property, we can write a 1D function as
$g(x) = \int_{-\infty}^{\infty} g(\xi)\delta(x-\xi)d\xi$. To gain intuition, consider the approximation
$g(x) \approx \sum_{n=-\infty}^{\infty} g(n\Delta x)\frac{1}{\Delta x}\Pi\left(\frac{x-n\Delta x}{\Delta x}\right)\Delta x.$



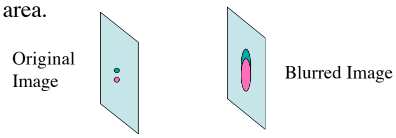
Representation of 2D Function

Similarly, we can write a 2D function as
$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x-\xi,y-\eta)d\xi d\eta.$
To gain intuition, consider the approximation
$g(x,y) \approx \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g(n\Delta x,m\Delta y)\frac{1}{\Delta x}\Pi\left(\frac{x-n\Delta x}{\Delta x}\right)\frac{1}{\Delta y}\Pi\left(\frac{y-m\Delta y}{\Delta y}\right)\Delta x\Delta y.$



Impulse Response

Intuition: the impulse response is the response of a system to an input of infinitesimal width and unit area.



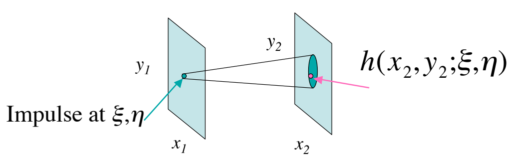
Since any input can be thought of as the weighted sum of impulses, a linear system is characterized by its impulse response(s).

Impulse Response

The impulse response characterizes the response of a system over all space to a Dirac delta impulse function at a certain location.

$h(x_2;\xi) = L[\delta(x_1-\xi)]$ 1D Impulse Response

$h(x_2,y_2;\xi,\eta) = L[\delta(x_1-\xi,y_1-\eta)]$ 2D Impulse Response



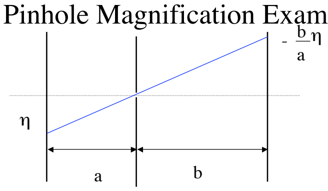
Superposition Integral

What is the response to an arbitrary function $g(x_1,y_1)$?

Write $g(x_1,y_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x_1-\xi,y_1-\eta)d\xi d\eta.$

The response is given by

Pinhole Magnification Example



In this example, an impulse at (ξ,η) will yield an impulse at $(M\xi, M\eta)$ where $M = -b/a$.

Thus, $h(x_2, y_2; \xi, \eta) = L[\delta(x_1 - \xi, y_1 - \eta)] = \delta(x_2 - M\xi, y_2 - M\eta)$.

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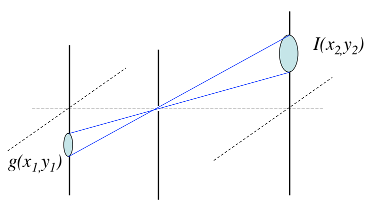
Pinhole Magnification Example

$$I(x_2,y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2,y_2;\xi,\eta)d\xi d\eta$$
$$= C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x_2 - M\xi, y_2 - M\eta)d\xi d\eta$$



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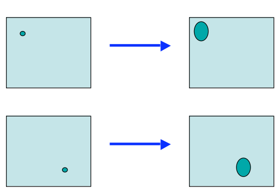
Space Invariance

If a system is space invariant, the impulse response depends only on the difference between the output coordinates and the position of the impulse and is given by $h(x_2,y_2;\xi,\eta) = h(x_2 - \xi, y_2 - \eta)$



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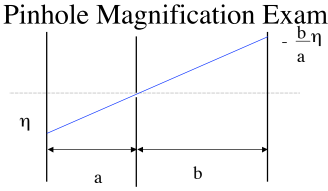
Pinhole Magnification Example

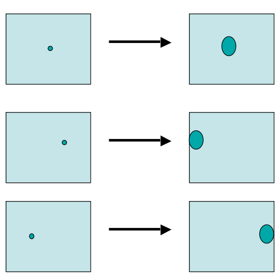


$h(x_2,y_2;\xi,\eta) = C\delta(x_2 - M\xi, y_2 - M\eta)$.

Is this system space invariant?

Pinhole Magnification Example

____, the pinhole system ____ space invariant.



2D Convolution

For a space invariant linear system, the superposition integral becomes a convolution integral.

$$I(x_2,y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta) h(x_2,y_2;\xi,\eta) d\xi d\eta$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta) h(x_2-\xi,y_2-\eta) d\xi d\eta$$
$$= g(x_2,y_2) ** h(x_2,y_2)$$

where ** denotes 2D convolution. This will sometimes be abbreviated as *, e.g. $I(x_2, y_2) = g(x_2, y_2) * h(x_2, y_2)$.

1D Convolution

For completeness, here is the 1D version.

$$I(x) = \int_{-\infty}^{\infty} g(\xi)h(x;\xi)d\xi$$
$$= \int_{-\infty}^{\infty} g(\xi)h(x-\xi)d\xi$$
$$= g(x) * h(x)$$

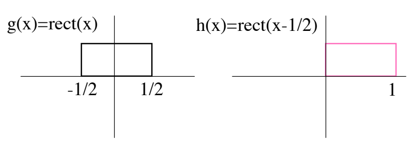
Useful fact:

$$g(x) * \delta(x-\Delta) = \int_{-\infty}^{\infty} g(\xi)\delta(x-\Delta-\xi)d\xi$$
$$= g(x-\Delta)$$

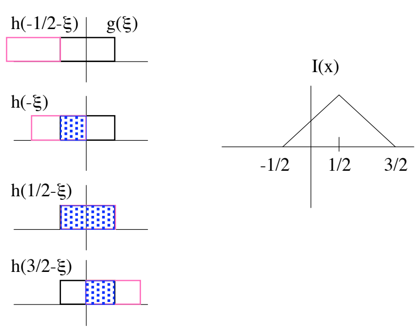
1D Convolution Review

$$g(x) * h(x) = \int_{-\infty}^{\infty} g(\xi)h(x-\xi)d\xi$$

Basic Rule: Flip one function, slide it past the other function, and integrate as you go.

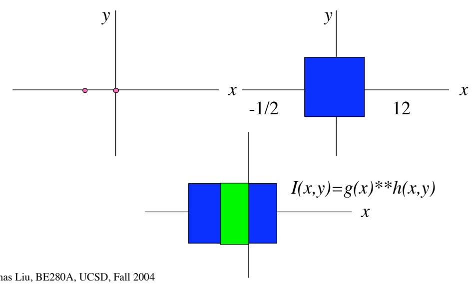


1D Convolution Review

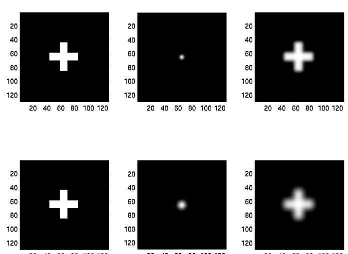


2D Convolution Example

$g(x) = \delta(x+1/2,y) + \delta(x,y)$ $\qquad$ $h(x)=rect(x,y)$



2D Convolution Example



Summary

1. The response to a linear system can be characterized by a spatially varying impulse response and the application of the superposition integral.
2. A shift invariant linear system can be characterized by its impulse response and the application of a convolution integral.
3. Dirac delta functions are generalized functions.

Pinhole Magnification Example

$$I(x_2,y_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(\xi,\eta)h(x_2,y_2;\xi,\eta)d\xi d\eta$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(\xi,\eta)\delta(x_2-M\xi,y_2-M\eta)d\xi d\eta$$

after substituting $\xi'=M\xi$ and $\eta'=M\eta$, we obtain

$$=\frac{1}{M^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(\xi'/M,\eta'/M)\delta(x_2-\xi',y_2-\eta')d\xi'd\eta'$$
$$=\frac{1}{M^2}g(x_2/M,y_2/M)**\delta(x_2,y_2)$$
$$=\frac{1}{M^2}g(x_2/M,y_2/M)$$

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