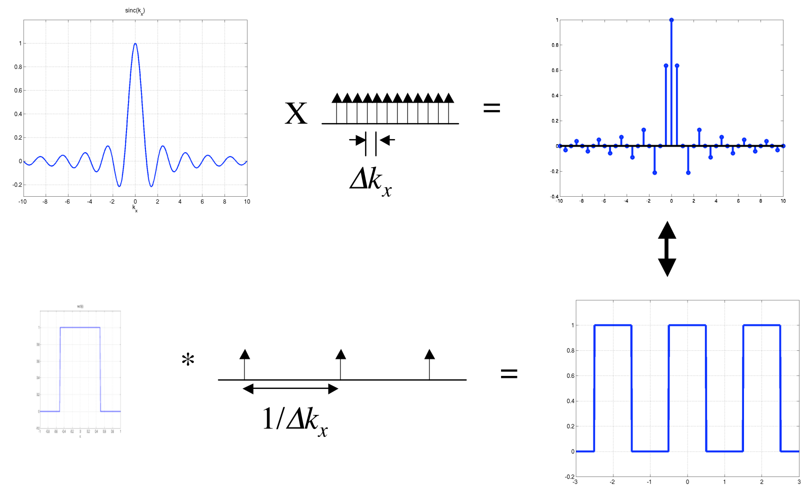
Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2004
Lecture 6
Resolution, Discrete Fourier Transform

Topics

1. Recap of sampling.
2. Resolution
3. Discrete Fourier Transform

Fourier Sampling -- Inverse Transform



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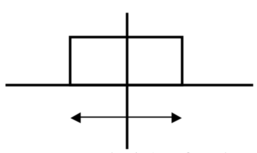
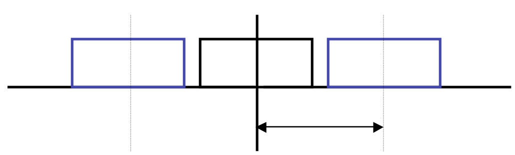
Nyquist Condition



FOV (Field of View)



$1/\Delta k_x$

To avoid overlap, $1/\Delta k_x >$ FOV, or equivalently, $\Delta k_x < 1/\text{FOV}$

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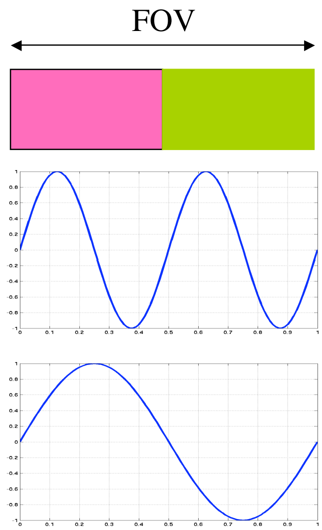
Intuitive view of Aliasing



$k_x = 2/FOV$

$k_x = 1/FOV$

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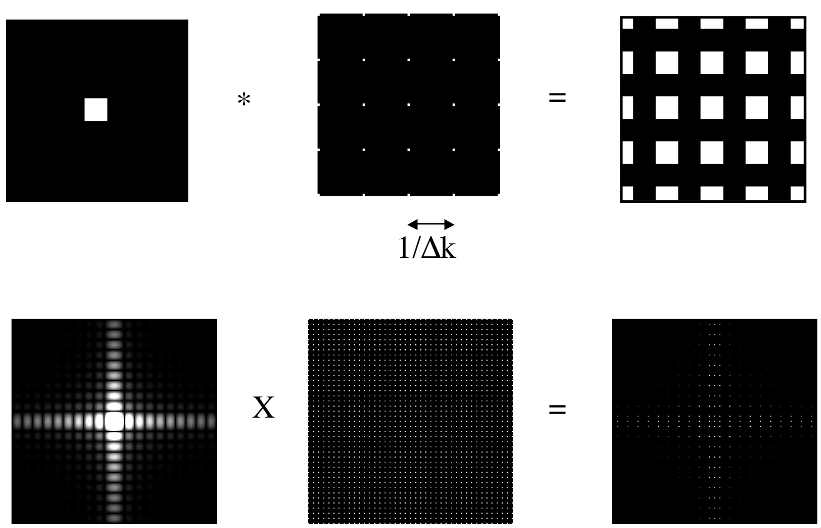

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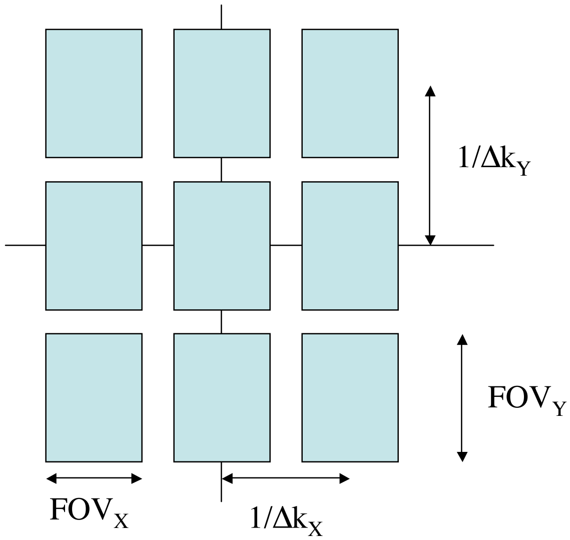
Nyquist Conditions



$1/\Delta k_Y > FOV_Y$

$1/\Delta k_X > FOV_X$

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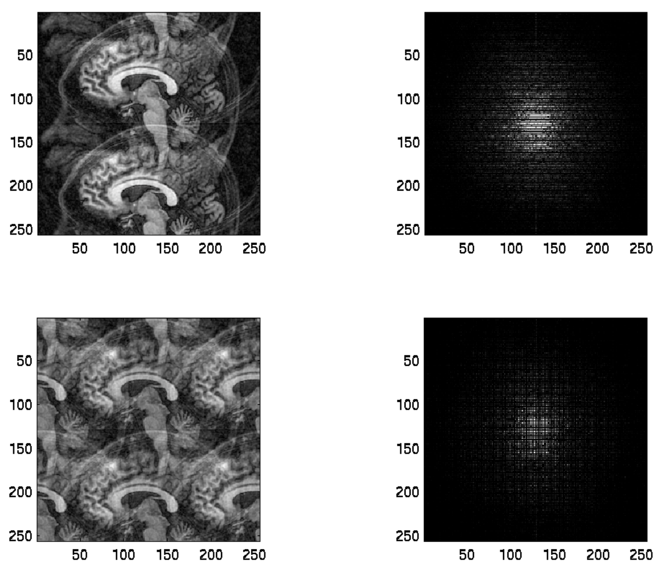
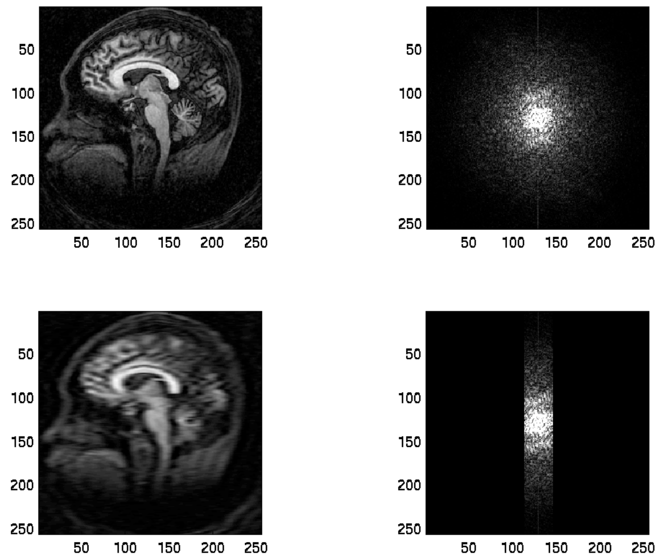
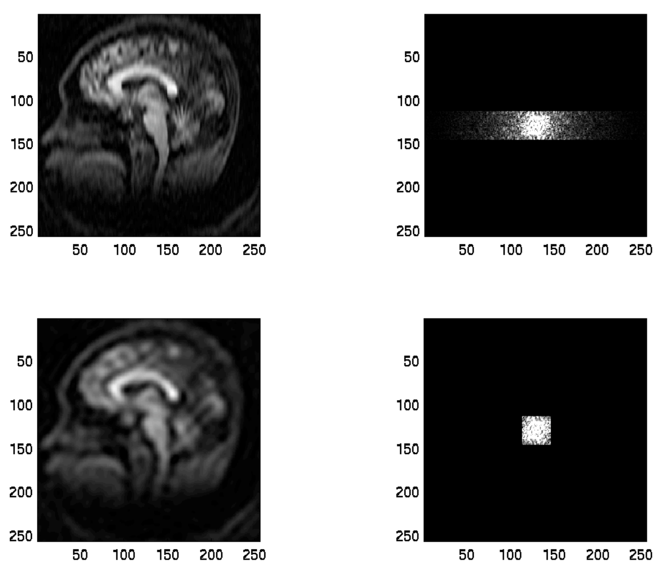
Aliasing



Resolution



Resolution



Windowing

Windowing the data in Fourier space

$$G_W(k_x,k_y) = G(k_x,k_y)W(k_x,k_y)$$

Results in convolution of the object with the inverse transform of the window

$$g_W(x,y) = g(x,y) * w(x,y)$$

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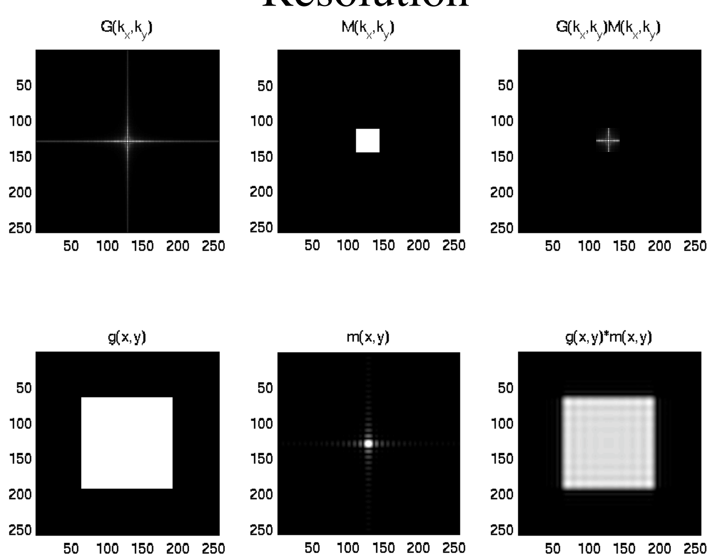
Resolution



Windowing Example

$$W(k_x,k_y) = rect\left(\frac{k_x}{W_{k_x}}\right)rect\left(\frac{k_y}{W_{k_y}}\right)$$

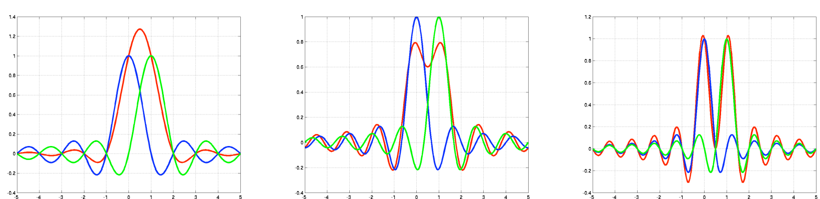
$$w(x,y) = F^{-1}\left[rect\left(\frac{k_x}{W_{k_x}}\right)rect\left(\frac{k_y}{W_{k_y}}\right)\right]$$
$$= W_{k_x}W_{k_y}\,\mathrm{sinc}\left(W_{k_x}x\right)\mathrm{sinc}\left(W_{k_y}y\right)$$

$$g_W(x,y) = g(x,y) ** W_{k_x}W_{k_y}\,\mathrm{sinc}\left(W_{k_x}x\right)\mathrm{sinc}\left(W_{k_y}y\right)$$

Windowing Example

$$g(x,y) = [\delta(x) + \delta(x-1)]\delta(y)$$

$$g_W(x,y) = [\delta(x) + \delta(x-1)]\delta(y) ** W_{k_x}W_{k_y}\,\mathrm{sinc}\left(W_{k_x}x\right)\mathrm{sinc}\left(W_{k_y}y\right)$$
$$= W_{k_x}W_{k_y}\left(\left[\delta(x) + \delta(x-1)\right] * \mathrm{sinc}\left(W_{k_x}x\right)\right)\mathrm{sinc}\left(W_{k_y}y\right)$$
$$= W_{k_x}W_{k_y}\left(\mathrm{sinc}\left(W_{k_x}x\right) + \mathrm{sinc}\left(W_{k_x}(x-1)\right)\right)\mathrm{sinc}\left(W_{k_y}y\right)$$



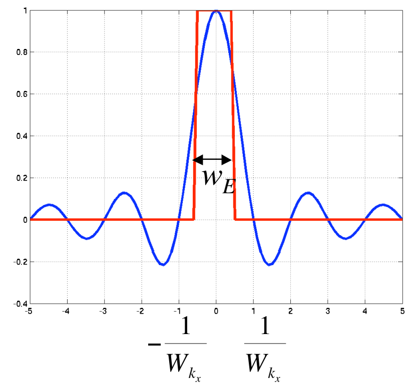
$W_{k_x} = 1$ $\quad W_{k_x} = 1.5$ $\quad W_{k_x} = 2$

Effective Width

$$w_E = \frac{1}{w(0)} \int_{-\infty}^{\infty} w(x)dx$$

Example

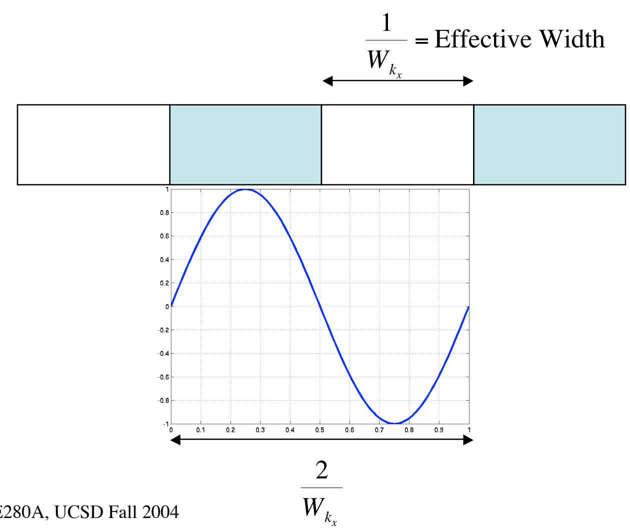
$$w_E = \frac{1}{1}\int_{-\infty}^{\infty} \mathrm{sinc}(W_{k_x} x)dx$$
$$= F\left[\mathrm{sinc}(W_{k_x} x)\right]\Big|_{k_x=0}$$
$$= \frac{1}{W_{k_x}} rect\left(\frac{k_x}{W_{k_x}}\right)\Bigg|_{k_x=0}$$
$$= \frac{1}{W_{k_x}}$$



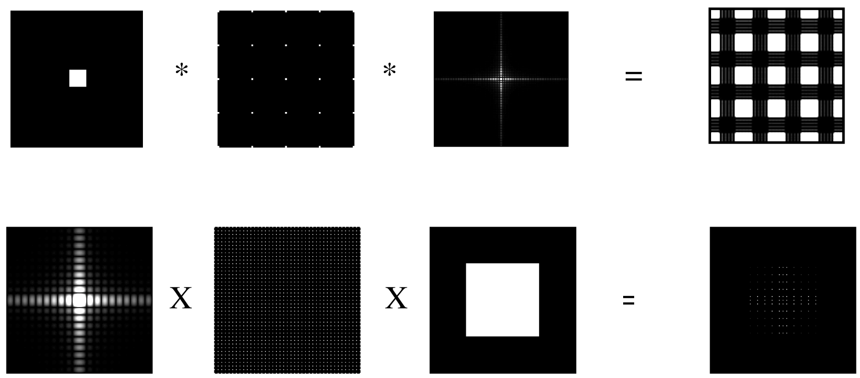
Resolution and spatial frequency

With a window of width W_{k_x} the highest spatial frequency is $W_{k_x}/2$.

This corresponds to a spatial period of $2/W_{k_x}$.



Sampling and Windowing



Sampling and Windowing

Sampling and windowing the data in Fourier space

$$G_{SW}\left(k_x,k_y\right) = G\left(k_x,k_y\right)\frac{1}{\Delta k_x \Delta k_y} comb\left(\frac{k_x}{\Delta k_x},\frac{k_y}{\Delta k_y}\right) rect\left(\frac{k_x}{W_{k_x}},\frac{k_y}{W_{k_y}}\right)$$

Results in replication and convolution in object space.

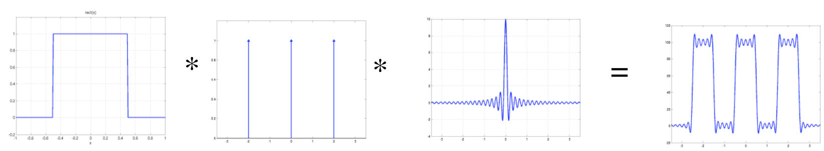
$$g_{SW}\left(x,y\right) = W_{k_x} W_{ky} g\left(x,y\right) ** comb(\Delta k_x x, \Delta k_y y) ** \operatorname{sinc}(W_{k_x} x)\operatorname{sinc}(W_{k_y} y)$$

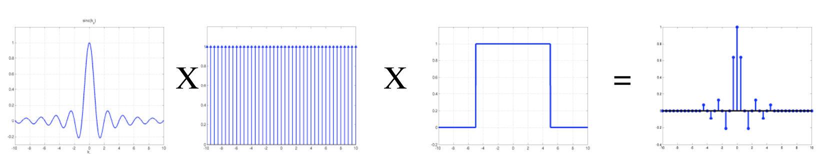
Discrete Fourier Transform

Idea: If we sample and window in the Fourier domain, we obtain a finite number of discrete Fourier samples. When we reconstruct the object, we should have the same number of pixels in our object.

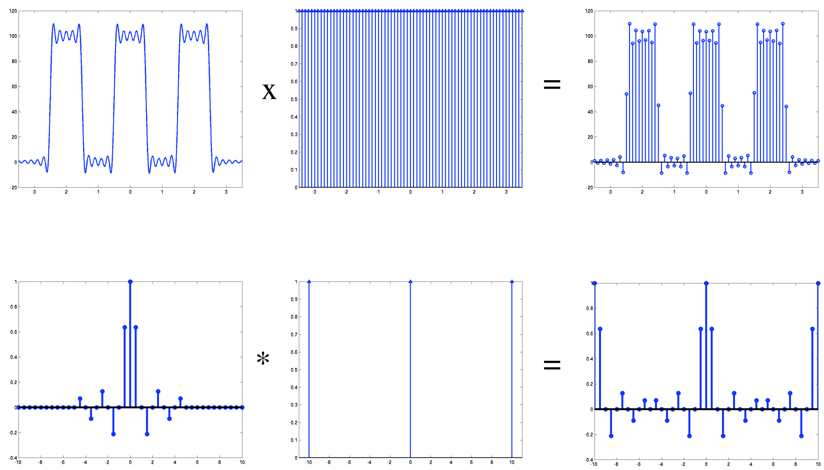
Also, the windowing process, has band-limited the sampled Fourier transform, so this allows us to sample the replicated object at discrete points.

1D Sampling and Windowing





Discrete Fourier Transform



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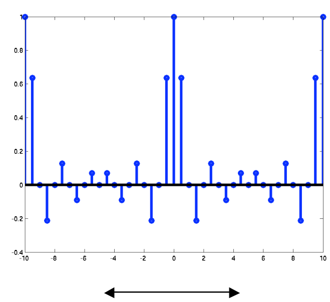
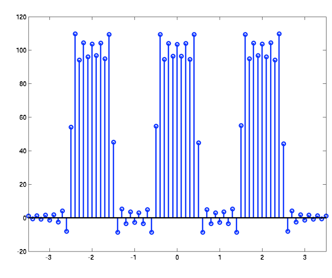
Discrete Fourier Transform



W_{k_x}



FOV

$$N_x = \frac{W_{k_x}}{\Delta k_x}$$

$$N_x = \frac{FOV_X}{1/W_{k_x}} = \frac{W_{k_x}}{\Delta k_x}$$

Note that $\frac{FOV_X}{N_x} = \frac{1}{W_{k_x}} = \delta_x$, our measure for resolution.

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DFT

Sampling, windowing, and replication in Fourier space

$$G_{DFT}(k_x,k_y) = G(k_x,k_y)\frac{1}{\Delta k_x \Delta k_y} comb\left(\frac{k_x}{\Delta k_x},\frac{k_y}{\Delta k_y}\right) rect\left(\frac{k_x}{W_{k_x}},\frac{k_y}{W_{k_y}}\right) ** comb\left(\frac{k_x}{W_{k_x}},\frac{k_y}{W_{k_y}}\right)$$

Replication, convolving, sampling in object space

$$g_{DFT}(x,y) = W_{k_x} W_{ky} g(x,y) ** comb(\Delta k_x x, \Delta k_y y) ** \mathrm{sinc}(W_{k_x} x)\mathrm{sinc}(W_{k_y} y) \\ \times W_{k_x} W_{ky} comb(W_{k_x} x, W_{k_x} y)$$

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DFT

$$\begin{aligned} F\left[g_D(x)\right] &= \int_0^{FOV} g_D(x) e^{-j2\pi m \Delta k_x x} dx \\ &= \int_0^{FOV} \sum_{n=-\infty}^{\infty} g_D(n/W_x)\delta(x - n/W_x) e^{-j2\pi m \Delta k_x x} dx \\ &= \sum_{n=0}^{N-1} \int_{-\infty}^{\infty} g_D(n/W_x)\delta(x - n/W_x) e^{-j2\pi m \Delta k_x x} dx \\ &= \sum_{n=0}^{N-1} g_D(n/W_x) e^{-j2\pi m n \Delta k_x / W_x} \\ &= \sum_{n=0}^{N-1} g_D[n] e^{-j2\pi m n / N} \end{aligned}$$

This is what MATLAB computes when you use fft

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DFT Basis Functions

$$\text{DFT}: \; G[m] = \sum_{n=0}^{N-1} g[n] e^{-j2\pi mn/N}$$

Basis Functions are therefore:

$$b_m[n] = e^{j2\pi mn/N}$$

Are these orthonormal??

$$\text{Inverse DFT}: \; g[n] = \frac{1}{N} \sum_{m=0}^{N-1} G[m] e^{j2\pi mn/N}$$

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2D DFT

$$\text{DFT}: \; G[r,s] = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} g[m,n] e^{-j2\pi(rm+sn)/N}$$

Basis Functions are therefore:

$$b_{r,s}[m,n] = e^{j2\pi(rm+sn)/N}$$

Are these orthonormal??

$$\text{Inverse DFT}: \; g[m,n] = \frac{1}{N^2} \sum_{r=0}^{N-1} \sum_{s=0}^{N-1} G[r,s] e^{j2\pi(rm+sn)/N}$$

In general, the number of points along each dimension need not be the same (e.g. $N_1 \neq N_2$). How does this change the expressions?

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