Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2006
CT/Fourier Lecture 2

Topics

- Modulation
- Modulation Transfer Function
- Convolution/Multiplication
- Revisit Projection-Slice Theorem
- Filtered Backprojection

Modulation

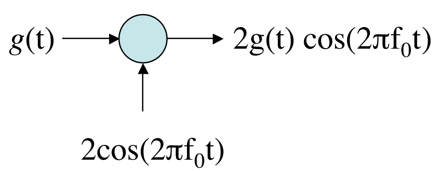
$$F\left[g(x)e^{j2\pi k_0 x}\right] = G(k_x) * \delta(k_x - k_0) = G\left(k_x - k_0\right)$$

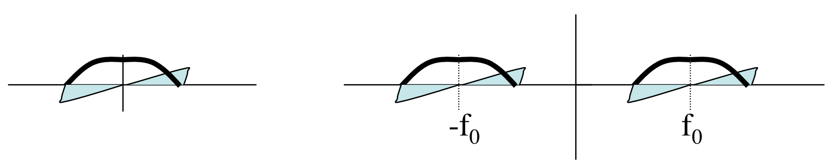
$$F\left[g(x)\cos\left(2\pi k_0 x\right)\right] = \frac{1}{2}G\left(k_x - k_0\right) + \frac{1}{2}G\left(k_x + k_0\right)$$

$$F\left[g(x)\sin\left(2\pi k_0 x\right)\right] = \frac{1}{2j}G\left(k_x - k_0\right) - \frac{1}{2j}G\left(k_x + k_0\right)$$

Example

Amplitude Modulation (e.g. AM Radio)

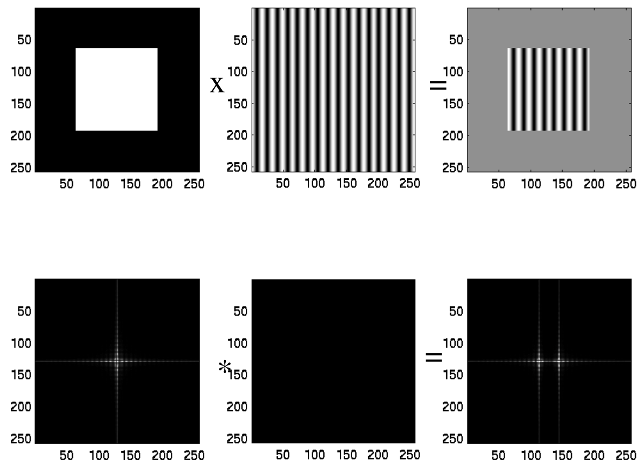




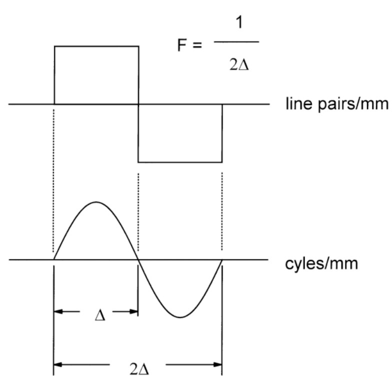
G(f)

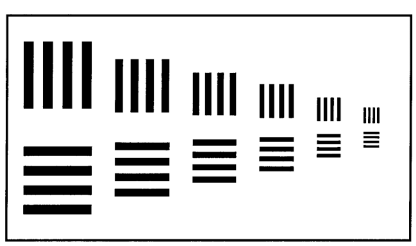
G(f-f$_0$)+ G(f+f$_0$)

Modulation Example

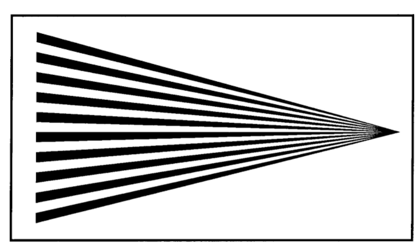


Bushberg et al 2001



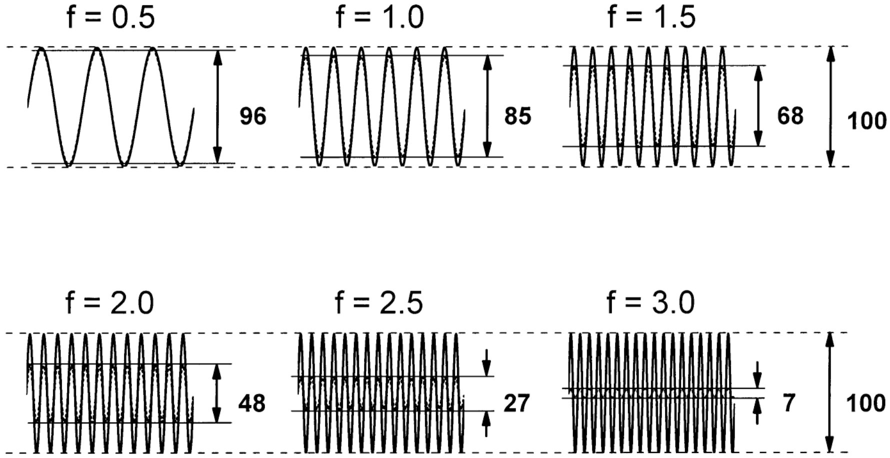


Line Pair Test Phantom



Section of a Star Pattern

Modulation Transfer Function (MTF)
or
Frequency Response



Bushberg et al 2001

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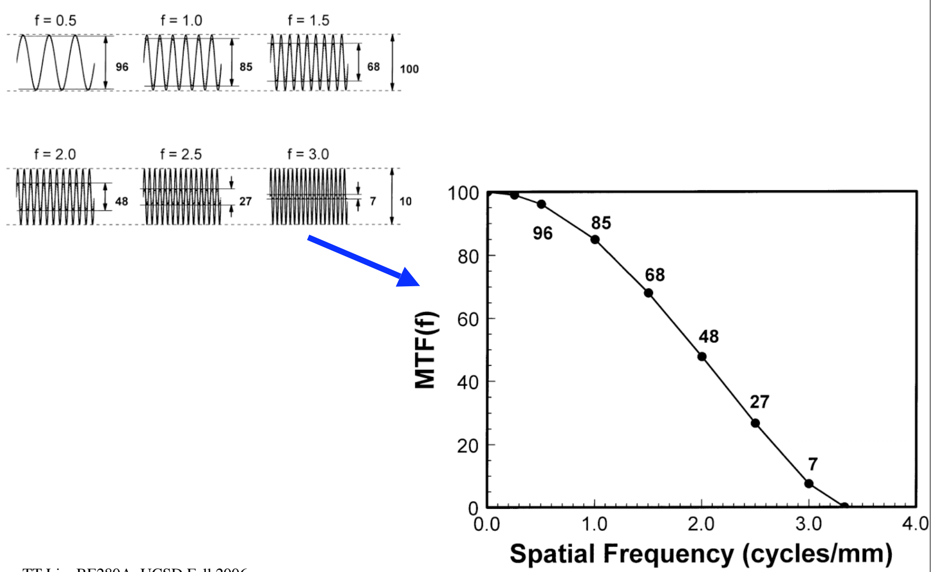
Modulation Transfer Function



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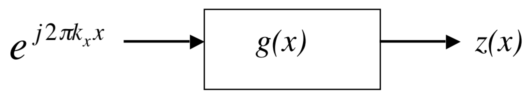
Bushberg et al 2001

Eigenfunctions

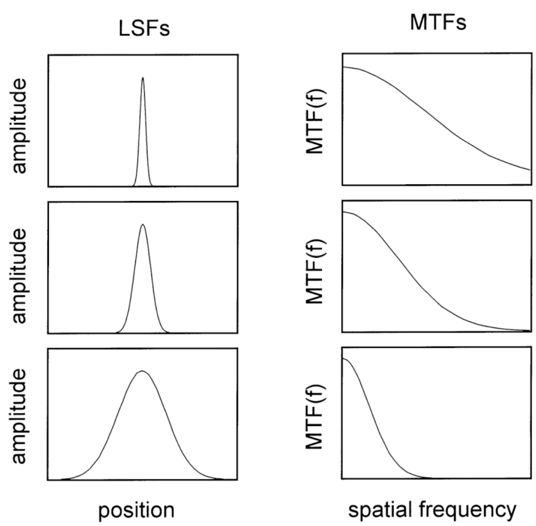
The fundamental nature of the convolution theorem may be better understood by observing that the complex exponentials are eigenfunctions of the convolution operator.



$$
\begin{aligned}
z(x) &= g(x) * e^{j2\pi k_x x} \\
&= \int_{-\infty}^{\infty} g(u) e^{j2\pi k_x (x-u)} du \\
&= G(k_x) e^{j2\pi k_x x}
\end{aligned}
$$

The response of a linear shift invariant system to a complex exponential is simply the exponential multiplied by the FT of the system's impulse response.

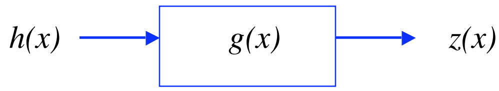
MTF = Fourier Transform of PSF



Bushberg et al 2001

Convolution/Multiplication

Now consider an arbitrary input *h(x)*.



Recall that we can express *h(x)* as the integral of weighted complex exponentials.

$$h(x) = \int_{-\infty}^{\infty} H(k_x) e^{j2\pi k_x x} dk_x$$

Each of these exponentials is weighted by $G(k_x)$ so that the response may be written as

$$z(x) = \int_{-\infty}^{\infty} G(k_x) H(k_x) e^{j2\pi k_x x} dk_x$$

Convolution/Modulation Theorem

$$\begin{aligned} F\{g(x) * h(x)\} &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g(u) * h(x-u) du \right] e^{-j2\pi k_x x} dx \\ &= \int_{-\infty}^{\infty} g(u) \int_{-\infty}^{\infty} h(x-u) e^{-j2\pi k_x x} dx du \\ &= \int_{-\infty}^{\infty} g(u) H(k_x) e^{-j2\pi k_x u} du \\ &= G(k_x) H(k_x) \end{aligned}$$

Convolution in the spatial domain transforms into multiplication in the frequency domain. Dual is modulation

$$F\{g(x)h(x)\} = G(k_x) * H(k_x)$$

2D Convolution/Multiplication

Convolution

$$F\left[g(x,y)**h(x,y)\right]=G(k_x,k_y)H(k_x,k_y)$$

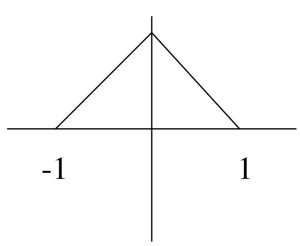
Multiplication

$$F\left[g(x,y)h(x,y)\right]=G(k_x,k_y)**H(k_x,k_y)$$

Application of Convolution Thm.

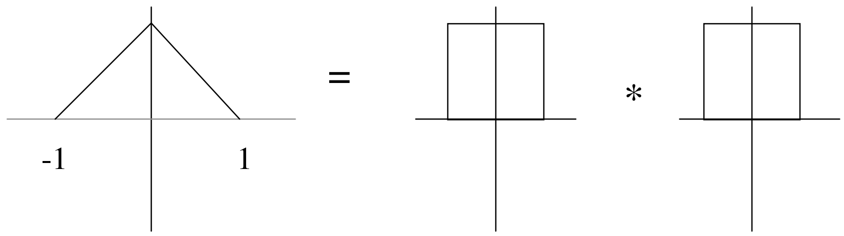
$$\Lambda(x)=\begin{cases}1-|x| & |x|<1\\ 0 & otherwise\end{cases}$$

$$F(\Lambda(x))=\int_{-1}^{1}(1-|x|)e^{-j2\pi k_x x}dx=??$$

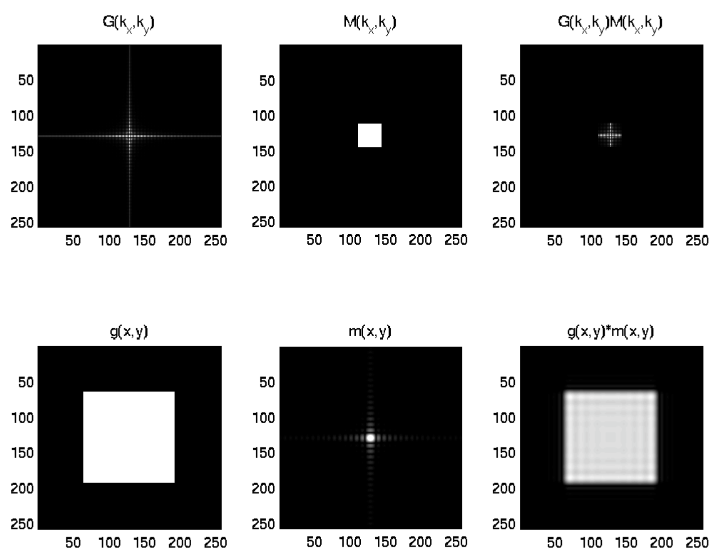


Application of Convolution Thm.

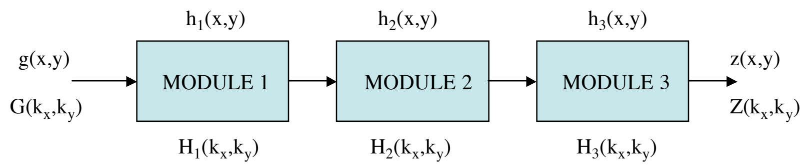
$$\Lambda(x) = \Pi(x) * \Pi(x)$$
$$F(\Lambda(x)) = \mathrm{sinc}^2(k_x)$$



Convolution Example



Response of an Imaging System



$$z(x,y)=g(x,y)**h_1(x,y)**h_2(x,y)**h_3(x,y)$$

$$Z(k_x,k_y)=G(k_x,k_y)\ H_1(k_x,k_y)\ H_2(k_x,k_y)\ H_3(k_x,k_y)$$

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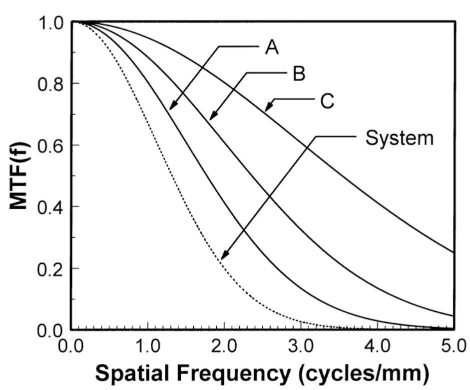
System MTF = Product of MTFs of Components



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Useful Approximation

$$FWHM_{System} = \sqrt{FWHM_1^2 + FWHM_2^2 + \cdots FWHM_N^2}$$

Example

$$FWHM_1 = 1mm$$

$$FWHM_2 = 2mm$$

$$FWHM_{system} = \sqrt{5} = 2.24mm$$

Figure 1:

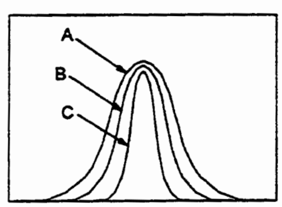
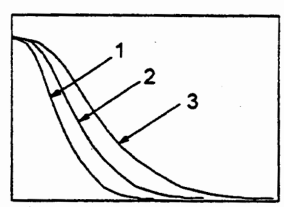


Figure 2:



8. Referring to Figure 1 (above) which demonstrates 3 different line spread functions (LSF), which LSF will yield the best spatial resolution?

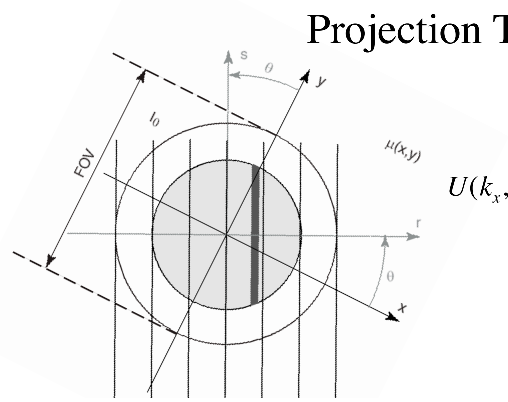
10. Referring to Figure 1 which shows LSFs, and Figure 2 which shows the corresponding modulation transfer functions (MTFs), which MTF corresponds to LSF C?

A. MTF number 1
B. MTF number 2
C. MTF number 3

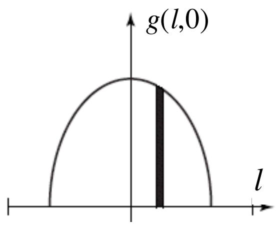
D74. The intrinsic resolution of a gamma camera is 5 mm. The collimator resolution is 10 mm. The overall system resolution is ______ mm.

A. 15
B. 11.2
C. 7.5
D. 5.0
E. 0.5

Projection Theorem



$$U(k_x,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu(x,y)e^{-j2\pi(k_x x + k_y y)}dxdy$$
$$= \int_{-\infty}^{-\infty}\left[\int_{-\infty}^{-\infty}\mu(x,y)dy\right]e^{-j2\pi k_x x}dx$$
$$= \int_{-\infty}^{-\infty}g(x,0)e^{-j2\pi k_x x}dx$$
$$= \int_{-\infty}^{-\infty}g(l,0)e^{-j2\pi kl}dl$$

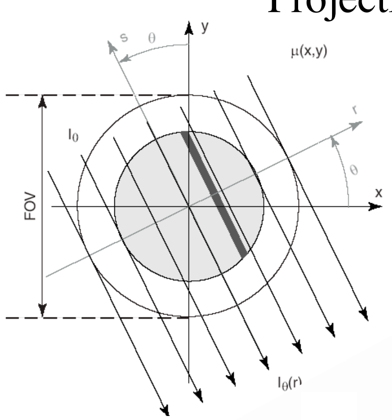


In-Class Example:

$$\mu(x,y) = \cos 2\pi x$$

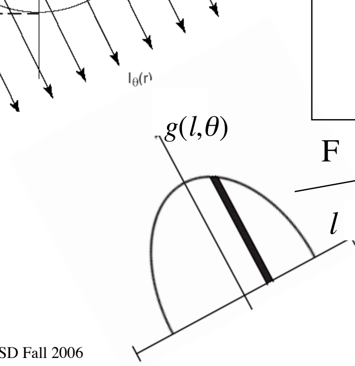
Suetens 2002

Projection Theorem



$$U(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu(x,y)e^{-j2\pi(k_x x + k_y y)}dxdy$$
$$= F_{2D}\left[\mu(x,y)\right]$$

$$U(k_x,k_y) = G(k,\theta)$$
$$k_x = k\cos\theta$$
$$k_y = k\sin\theta$$
$$k = \sqrt{k_x^2 + k_y^2}$$



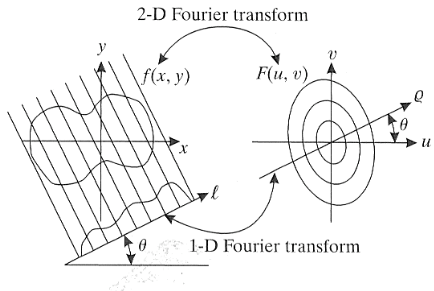
F →

$$G(k,\theta) = \int_{-\infty}^{\infty}g(l,\theta)e^{-j2\pi kl}dl$$

Suetens 2002

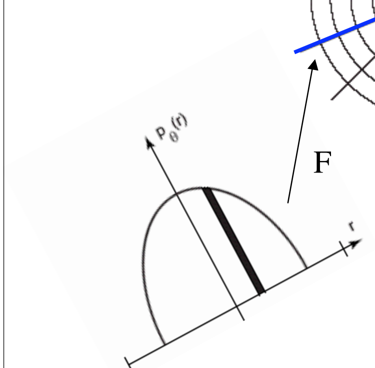
Projection Slice Theorem

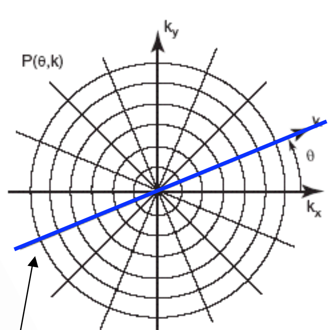
$$
\begin{aligned}
G(\rho,\theta) &= \int_{-\infty}^{\infty} g(l,\theta) e^{-j2\pi\rho l} dl \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - l) e^{-j2\pi\rho l} dx\, dy\, dl \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi\rho(x\cos\theta + y\sin\theta)} dx\, dy \\
&= F_{2D}\left[f(x,y)\right]\Big|_{u=\rho\cos\theta, v=\rho\sin\theta}
\end{aligned}
$$

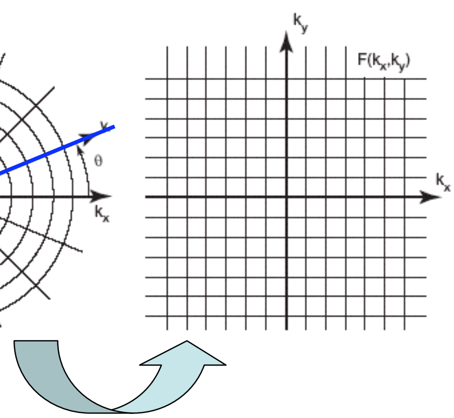


Prince&Links 2006

Fourier Reconstruction



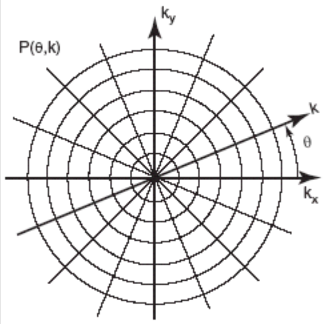




Interpolate onto Cartesian grid then take inverse transform

Suetens 2002

Polar Version of Inverse FT



$$\mu(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x,k_y)e^{j2\pi(k_x x + k_y y)}dk_x dk_y$$
$$= \int_0^{2\pi}\int_0^{\infty} G(k,\theta)e^{j2\pi(xk\cos\theta + yk\sin\theta)}kdkd\theta$$
$$= \int_0^{\pi}\int_{-\infty}^{\infty} G(k,\theta)e^{j2\pi k(x\cos\theta + y\sin\theta)}|k|dkd\theta$$

Note :

$g(l,\theta+\pi) = g(-l,\theta)$

So

$G(k,\theta+\pi) = G(-k,\theta)$

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Suetens 2002

Filtered Backprojection

$$\mu(x,y) = \int_0^{\pi}\int_{-\infty}^{\infty} G(k,\theta)e^{j2\pi(xk\cos\theta + yk\sin\theta)}|k|dkd\theta$$
$$= \int_0^{\pi}\int_{-\infty}^{\infty} |k|G(k,\theta)e^{j2\pi kl}dkd\theta$$
$$= \int_0^{\pi} g^*(l,\theta)d\theta$$

← Backproject a filtered projection

where $l = x\cos\theta + y\sin\theta$

$$g^*(l,\theta) = \int_{-\infty}^{\infty} |k|G(k,\theta)e^{j2\pi kl}dk$$
$$= g(l,\theta) * F^{-1}\left[|k|\right]$$
$$= g(l,\theta) * q(l)$$

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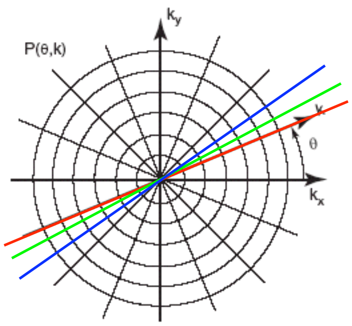
Fourier Interpretation

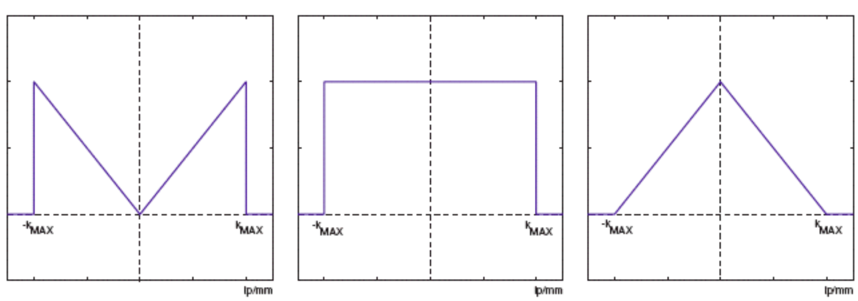


$$\text{Density} \approx \frac{N}{\text{circumference}} \approx \frac{N}{2\pi|k|}$$

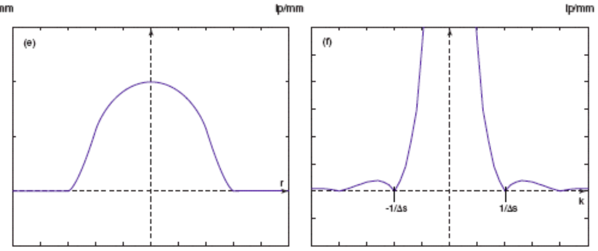
Low frequencies are oversampled. So to compensate for this, multiply the k-space data by |k| before inverse transforming.

Kak and Slaney; Suetens 2002

Ram-Lak Filter



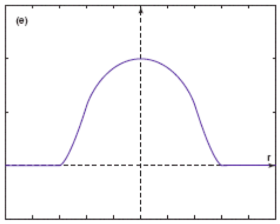
$k_{max}=1/\Delta s$



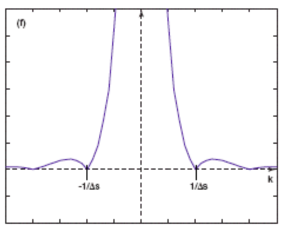
Suetens 2002

Reconstruction Path

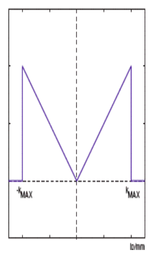
Projection



F ↓



x

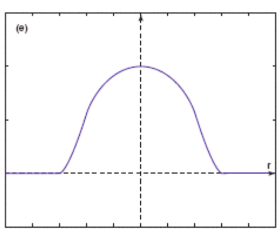


F^{-1} → Filtered Projection → Back-Project

Suetens 2002

Reconstruction Path

Projection



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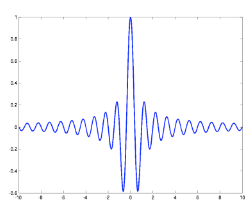


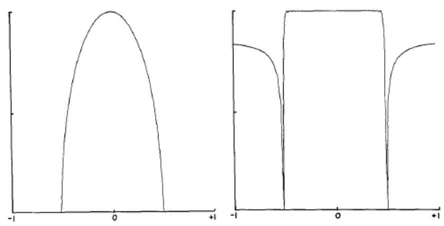
→ Filtered Projection

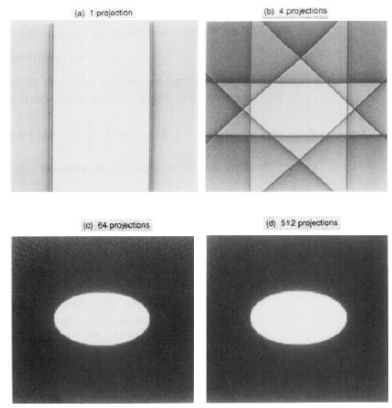
↓

Back-Project

Suetens 2002

Example





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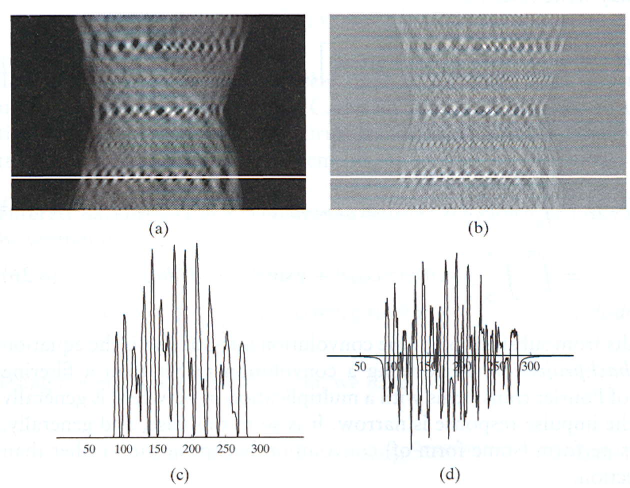
Kak and Slaney

Example

Figure 6.15
Convolution step:
(a) Original sinogram;
(b) filtered sinogram;
(c) profile of sinogram row [white line in (a)]; and
(d) profile of filtered sinogram row [white line in (b)].



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Prince and Links 2005

Example

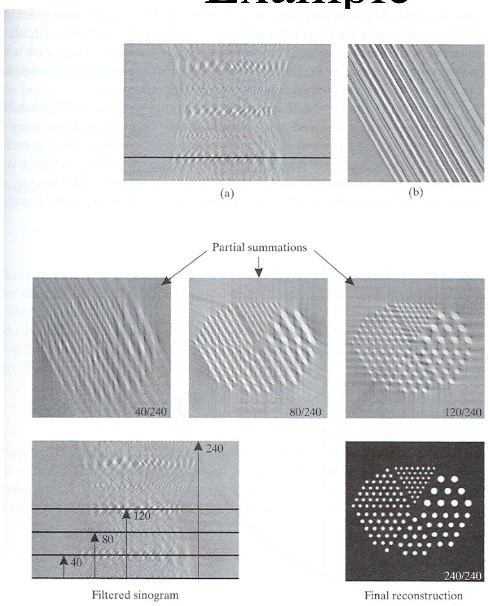

Figure 6.16
Backprojection step.

Figure 6.17
Summation step.

Prince and Links 2005