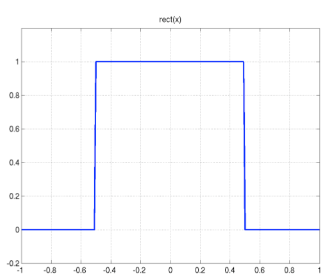
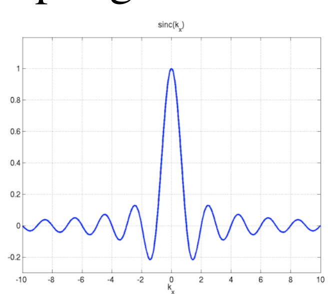
Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2006
MRI Lecture 3

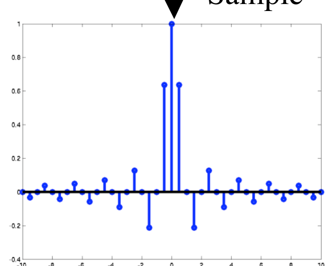
Fourier Sampling



→ F



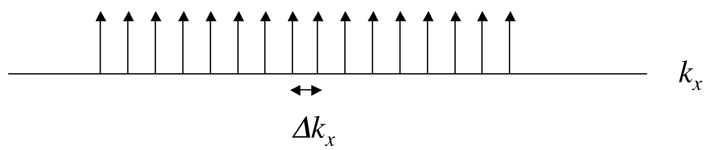
↓ Sample



Instead of sampling the signal, we sample its Fourier Transform

??? ← F^{-1}

Fourier Sampling

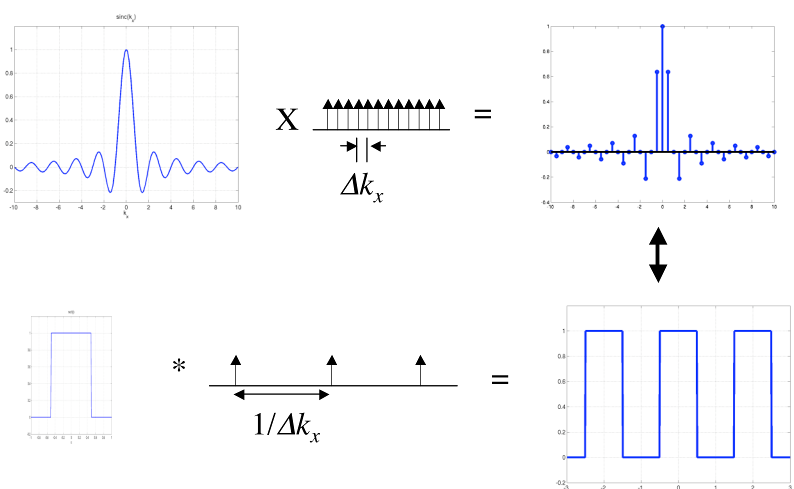


$$G_S(k_x) = G(k_x)\frac{1}{\Delta k_x} comb\left(\frac{k_x}{\Delta k_x}\right)$$

$$= G(k_x)\sum_{n=-\infty}^{\infty}\delta(k_x - n\Delta k_x)$$

$$= \sum_{n=-\infty}^{\infty} G(n\Delta k_x)\delta(k_x - n\Delta k_x)$$

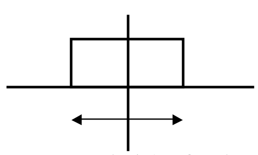
Fourier Sampling -- Inverse Transform



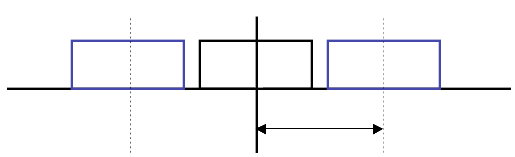
Fourier Sampling -- Inverse Transform

$$
\begin{aligned}
g_S(x) &= F^{-1}\left[G_S(k_x)\right] \\
&= F^{-1}\left[G(k_x)\frac{1}{\Delta k_x} comb\left(\frac{k_x}{\Delta k_x}\right)\right] \\
&= F^{-1}\left[G(k_x)\right] * F^{-1}\left[\frac{1}{\Delta k_x} comb\left(\frac{k_x}{\Delta k_x}\right)\right] \\
&= g(x) * comb\left(x\Delta k_x\right) \\
&= g(x) * \sum_{n=-\infty}^{\infty} \delta(x\Delta k_x - n) \\
&= g(x) * \frac{1}{\Delta k_x} \sum_{n=-\infty}^{\infty} \delta(x - \frac{n}{\Delta k_x}) \\
&= \frac{1}{\Delta k_x} \sum_{n=-\infty}^{\infty} g(x - \frac{n}{\Delta k_x})
\end{aligned}
$$

Nyquist Condition



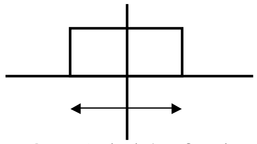
FOV (Field of View)

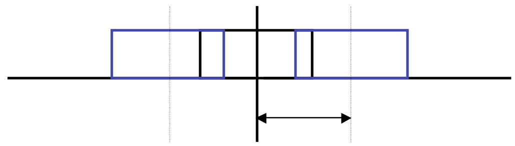


$1/\Delta k_x$

To avoid overlap, $1/\Delta k_x >$ FOV, or equivalently, $\Delta k_x < 1/\text{FOV}$

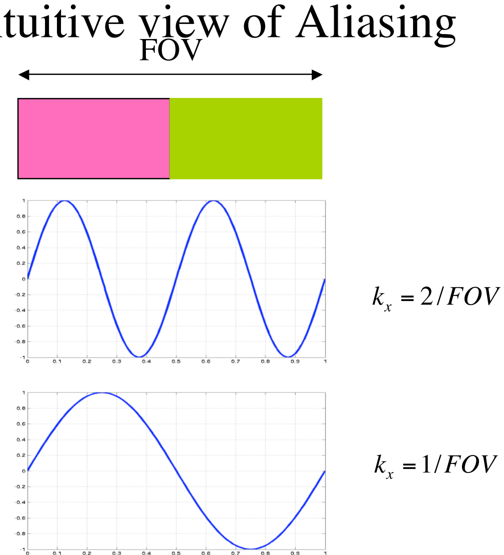
Aliasing



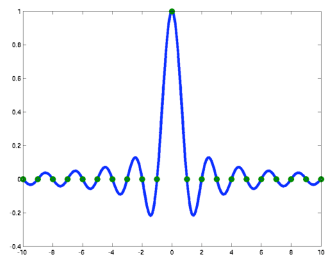


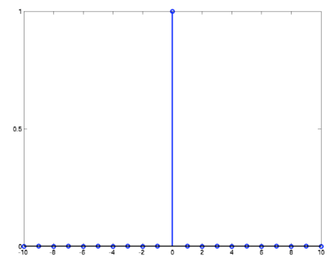
Aliasing occurs when $1/\Delta k_x <$ FOV

Intuitive view of Aliasing

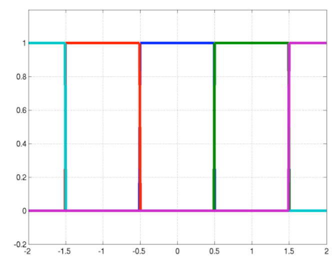


Aliasing Example





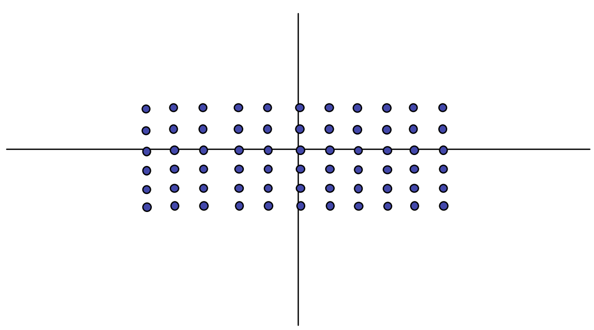
$\Delta k_x = 1$



$1/\Delta k_x = 1$

2D Comb Function

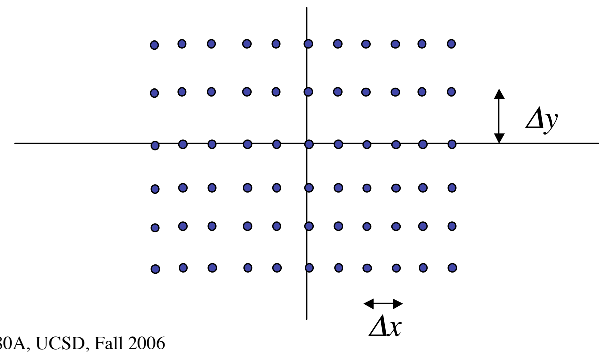
$$
\begin{aligned}
comb(x,y) &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x-m, y-n) \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x-m)\delta(y-n) \\
&= comb(x)comb(y)
\end{aligned}
$$



Scaled 2D Comb Function

$$comb(x/\Delta x, y/\Delta y) = comb(x/\Delta x)comb(y/\Delta y)$$

$$= \Delta x \Delta y \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x)\delta(y - n\Delta y)$$



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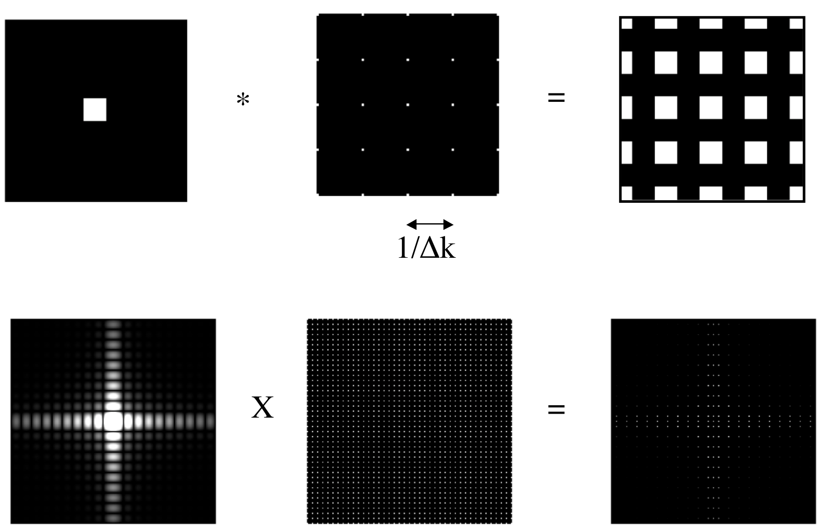

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2D k-space sampling

$$G_S(k_x,k_y) = G(k_x,k_y)\frac{1}{\Delta k_x \Delta k_y} comb\left(\frac{k_x}{\Delta k_x},\frac{k_y}{\Delta k_y}\right)$$

$$= G(k_x,k_y)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(k_x - m\Delta k_x, k_y - n\Delta k_y)$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} G(m\Delta k_x, n\Delta k_y)\delta(k_x - m\Delta k_x, k_y - n\Delta k_y)$$

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2D k-space sampling

$$g_S(x,y) = F^{-1}\left[G_S(k_x,k_y)\right]$$

$$= F^{-1}\left[G(k_x,k_y)\frac{1}{\Delta k_x \Delta k_y} comb\left(\frac{k_x}{\Delta k_x},\frac{k_y}{\Delta k_y}\right)\right]$$

$$= F^{-1}\left[G(k_x,k_y)\right] * F^{-1}\left[\frac{1}{\Delta k_x \Delta k_y} comb\left(\frac{k_x}{\Delta k_x},\frac{k_y}{\Delta k_y}\right)\right]$$

$$= g(x,y) ** comb\left(x\Delta k_x\right) comb\left(y \Delta k_y\right)$$

$$= g(x) ** \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x\Delta k_x - m)\delta(y\Delta k_y - n)$$

$$= g(x) ** \frac{1}{\Delta k_x \Delta k_y}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x - \frac{m}{\Delta k_x})\delta(y - \frac{n}{\Delta k_y})$$

$$= \frac{1}{\Delta k_x \Delta k_y}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g(x - \frac{m}{\Delta k_x}, y - \frac{n}{\Delta k_y})$$

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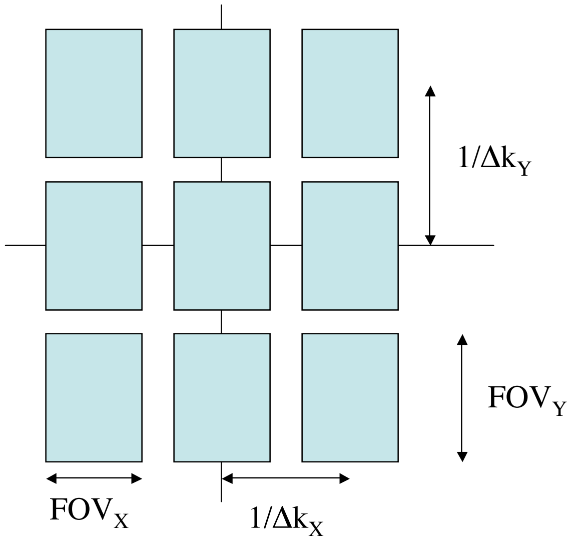
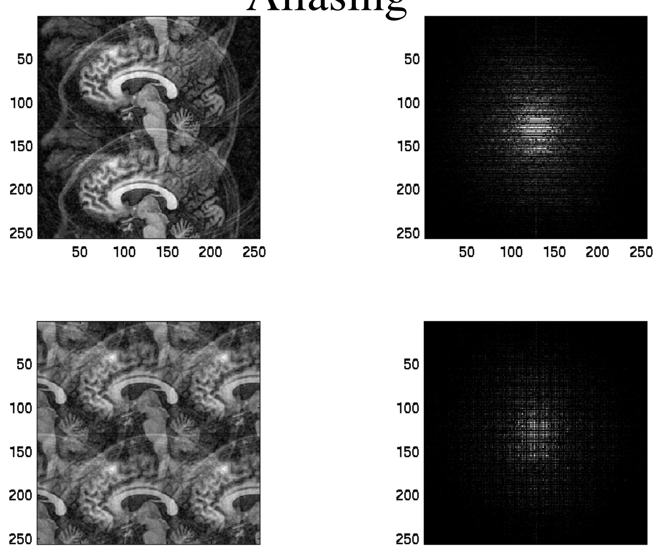
Nyquist Conditions



$1/\Delta k_Y > FOV_Y$

$1/\Delta k_X > FOV_X$

Aliasing



Windowing

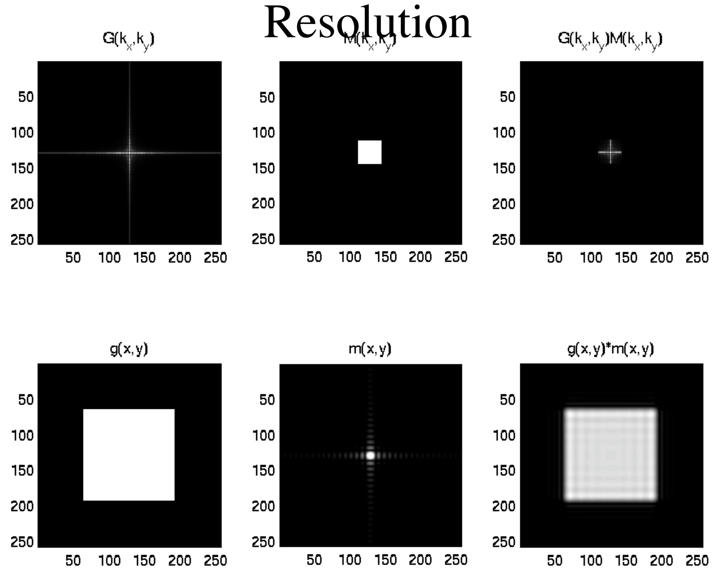
Windowing the data in Fourier space

$$G_W\left(k_x,k_y\right) = G\left(k_x,k_y\right)W\left(k_x,k_y\right)$$

Results in convolution of the object with the inverse transform of the window

$$g_W\left(x,y\right) = g\left(x,y\right) * w(x,y)$$

Resolution



Windowing Example

$$W\left(k_x,k_y\right) = rect\left(\frac{k_x}{W_{k_x}}\right)rect\left(\frac{k_y}{W_{k_y}}\right)$$

$$w\left(x,y\right) = F^{-1}\left[rect\left(\frac{k_x}{W_{k_x}}\right)rect\left(\frac{k_y}{W_{k_y}}\right)\right]$$

$$= W_{k_x}W_{k_y}\ \mathrm{sinc}\left(W_{k_x}x\right)\mathrm{sinc}\left(W_{k_y}y\right)$$

$$g_W\left(x,y\right) = g(x,y) ** W_{k_x}W_{k_y}\ \mathrm{sinc}\left(W_{k_x}x\right)\mathrm{sinc}\left(W_{k_y}y\right)$$

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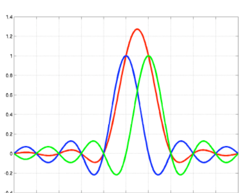
Windowing Example

$$g(x,y) = \left[\delta(x)+\delta(x-1)\right]\delta(y)$$

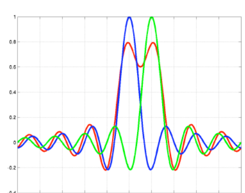
$$g_W\left(x,y\right) = \left[\delta(x)+\delta(x-1)\right]\delta(y) ** W_{k_x}W_{k_y}\ \mathrm{sinc}\left(W_{k_x}x\right)\mathrm{sinc}\left(W_{k_y}y\right)$$

$$= W_{k_x}W_{k_y}\left(\left[\delta(x)+\delta(x-1)\right] * \mathrm{sinc}\left(W_{k_x}x\right)\right)\mathrm{sinc}\left(W_{k_y}y\right)$$

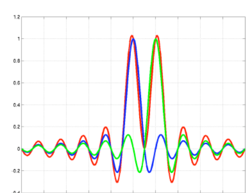
$$= W_{k_x}W_{k_y}\left(\mathrm{sinc}\left(W_{k_x}x\right)+\mathrm{sinc}\left(W_{k_x}(x-1)\right)\right)\mathrm{sinc}\left(W_{k_y}y\right)$$



$W_{k_x} = 1$



$W_{k_x} = 1.5$



$W_{k_x} = 2$

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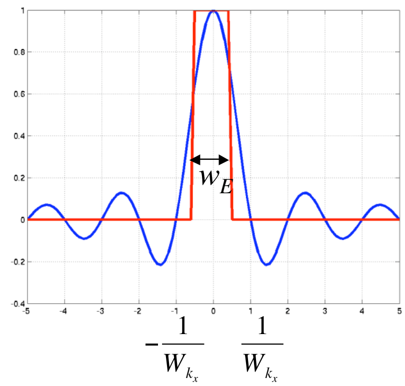
Effective Width

$$w_E = \frac{1}{w(0)} \int_{-\infty}^{\infty} w(x)dx$$

Example

$$\begin{aligned} w_E &= \frac{1}{1}\int_{-\infty}^{\infty} \operatorname{sinc}(W_{k_x}x)dx \\ &= F\left[\operatorname{sinc}(W_{k_x}x)\right]\Big|_{k_x=0} \\ &= \frac{1}{W_{k_x}} rect\left(\frac{k_x}{W_{k_x}}\right)\Bigg|_{k_x=0} \\ &= \frac{1}{W_{k_x}} \end{aligned}$$



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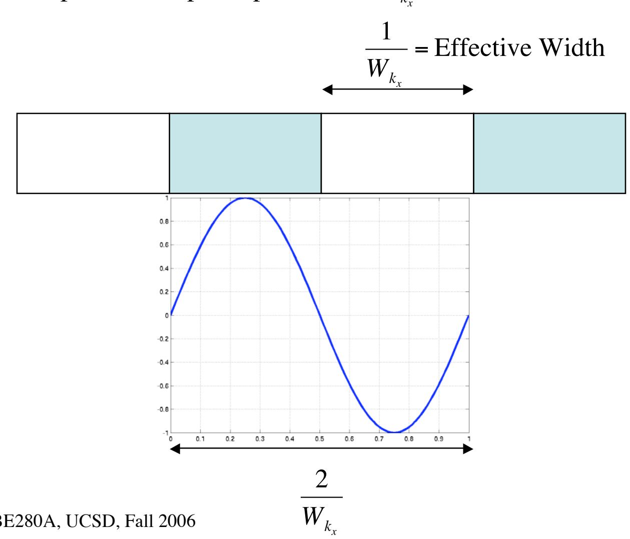
Resolution and spatial frequency

With a window of width W_{k_x} the highest spatial frequency is $W_{k_x}/2$.

This corresponds to a spatial period of $2/W_{k_x}$.



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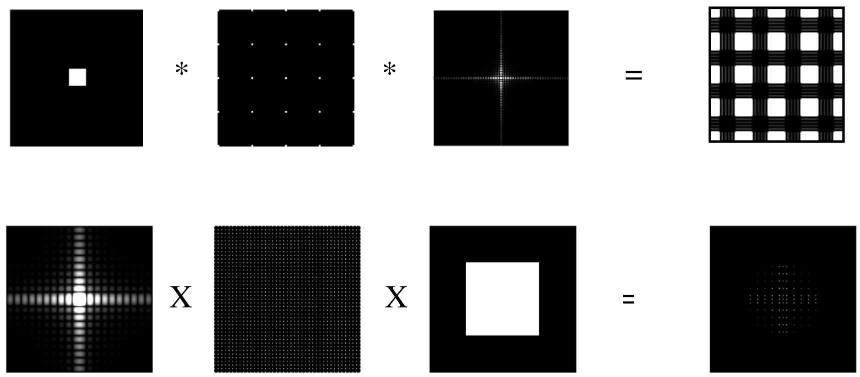
Sampling and Windowing



Sampling and Windowing

Sampling and windowing the data in Fourier space

$$G_{SW}\left(k_x,k_y\right) = G\left(k_x,k_y\right)\frac{1}{\Delta k_x \Delta k_y} comb\left(\frac{k_x}{\Delta k_x},\frac{k_y}{\Delta k_y}\right) rect\left(\frac{k_x}{W_{k_x}},\frac{k_y}{W_{k_y}}\right)$$

Results in replication and convolution in object space.

$$g_{SW}\left(x,y\right) = W_{k_x} W_{ky} g\left(x,y\right) ** comb(\Delta k_x x, \Delta k_y y) ** \mathrm{sinc}(W_{k_x} x)\,\mathrm{sinc}(W_{k_y} y)$$