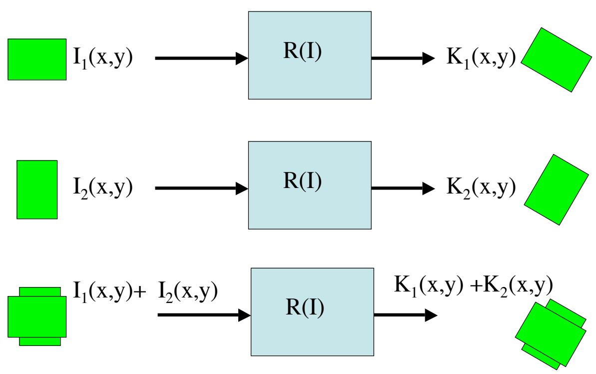
Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2007
X-Rays Lecture 3

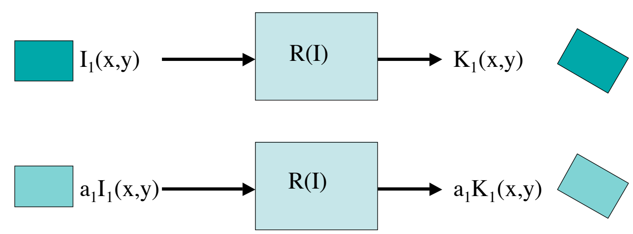
Topics

- Linearity
- Superposition
- Convolution

Linearity (Addition)



Linearity (Scaling)



Linearity

A system R is linear if for two inputs $I_1(x,y)$ and $I_2(x,y)$ with outputs

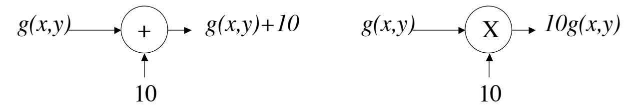
$R(I_1(x,y))=K_1(x,y)$ and $R(I_2(x,y))=K_2(x,y)$

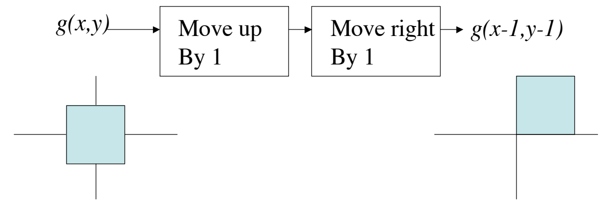
the response to the weighted sum of inputs is the weighted sum of outputs:

$R(a_1 I_1(x,y)+ a_2 I_2(x,y))=a_1 K_1(x,y)+ a_2 K_2(x,y)$

Example

Are these linear systems?





Superposition

$$g[m] = g[0]\delta[m] + g[1]\delta[m-1] + g[2]\delta[m-2]$$

$$h[m',k] = L[\delta[m-k]]$$

$$\begin{aligned} y[m'] &= L\big[g[m]\big] \\ &= L\big[g[0]\delta[m] + g[1]\delta[m-1] + g[2]\delta[m-2]\big] \\ &= L\big[g[0]\delta[m]\big] + L\big[g[1]\delta[m-1]\big] + L\big[g[2]\delta[m-2]\big] \\ &= g[0]L\big[\delta[m]\big] + g[1]L\big[\delta[m-1]\big] + g[2]L\big[\delta[m-2]\big] \\ &= g[0]h[m',0] + g[1]h[m',1] + g[2]h[m',2] \\ &= \sum_{k=0}^{2} g[k]h[m',k] \end{aligned}$$

Superposition Integral

What is the response to an arbitrary function $g(x_1,y_1)$?

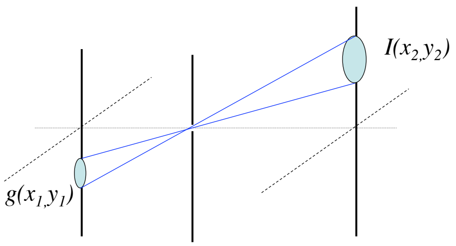
Write $g(x_1,y_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x_1-\xi,y_1-\eta)d\xi d\eta.$

The response is given by

$$\begin{aligned} I(x_2,y_2) &= L\big[g_1(x_1,y_1)\big] \\ &= L\Big[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x_1-\xi,y_1-\eta)d\xi d\eta\Big] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)L\big[\delta(x_1-\xi,y_1-\eta)\big]d\xi d\eta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2,y_2;\xi,\eta)d\xi d\eta \end{aligned}$$

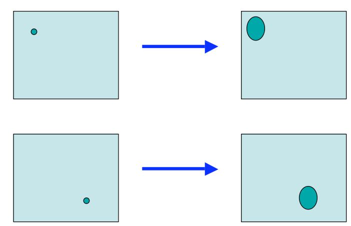
Pinhole Magnification Example

$$I(x_2,y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2,y_2;\xi,\eta)d\xi d\eta$$
$$= C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x_2 - m\xi, y_2 - m\eta)d\xi d\eta$$

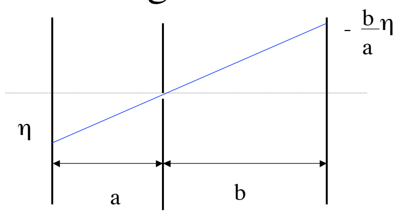


Space Invariance

If a system is space invariant, the impulse response depends only on the difference between the output coordinates and the position of the impulse and is given by $h(x_2,y_2;\xi,\eta) = h(x_2 - \xi, y_2 - \eta)$



Pinhole Magnification Example

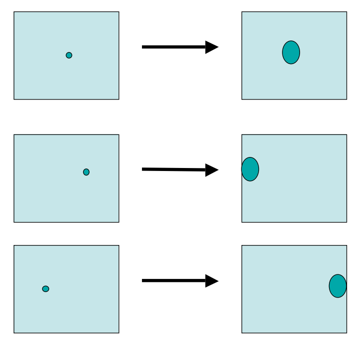


$h(x_2,y_2;\xi,\eta) = C\delta(x_2 - m\xi, y_2 - m\eta)$.

Is this system space invariant?

Pinhole Magnification Example

____, the pinhole system ____ space invariant.



Convolution

$$g[m] = g[0]\delta[m] + g[1]\delta[m-1] + g[2]\delta[m-2]$$

$$h[m',k] = L[\delta[m-k]] = h[m-k]$$

$$\begin{aligned} y[m'] &= L\big[g[m]\big] \\ &= L\big[g[0]\delta[m] + g[1]\delta[m-1] + g[2]\delta[m-2]\big] \\ &= L\big[g[0]\delta[m]\big] + L\big[g[1]\delta[m-1]\big] + L\big[g[2]\delta[m-2]\big] \\ &= g[0]L\big[\delta[m]\big] + g[1]L\big[\delta[m-1]\big] + g[2]L\big[\delta[m-2]\big] \\ &= g[0]h[m'-0] + g[1]h[m'-1] + g[2]h[m'-2] \\ &= \sum_{k=0}^{2} g[k]h[m'-k] \end{aligned}$$

1D Convolution

$$\begin{aligned} I(x) &= \int_{-\infty}^{\infty} g(\xi)h(x;\xi)d\xi \\ &= \int_{-\infty}^{\infty} g(\xi)h(x-\xi)d\xi \\ &= g(x) * h(x) \end{aligned}$$

Useful fact:

$$\begin{aligned} g(x) * \delta(x-\Delta) &= \int_{-\infty}^{\infty} g(\xi)\delta(x-\Delta-\xi)d\xi \\ &= g(x-\Delta) \end{aligned}$$

2D Convolution

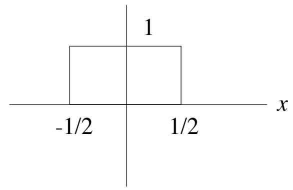
For a space invariant linear system, the superposition integral becomes a convolution integral.

$$\begin{aligned} I(x_2,y_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2,y_2;\xi,\eta)d\xi d\eta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2-\xi,y_2-\eta)d\xi d\eta \\ &= g(x_2,y_2) ** h(x_2,y_2) \end{aligned}$$

where ** denotes 2D convolution. This will sometimes be abbreviated as *, e.g. I$(x_2, y_2) = g(x_2, y_2) * h(x_2, y_2)$.

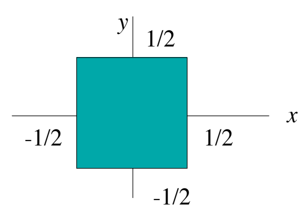
Rectangle Function

$$\Pi(x) = \begin{cases} 0 & |x| > 1/2 \\ 1 & |x| \le 1/2 \end{cases}$$

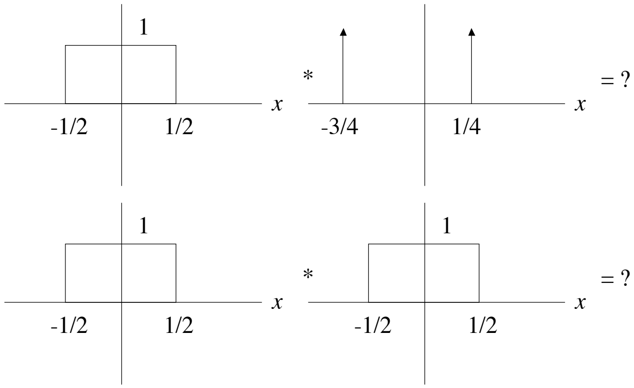


Also called rect(x)

$$\Pi(x,y) = \Pi(x)\Pi(y)$$

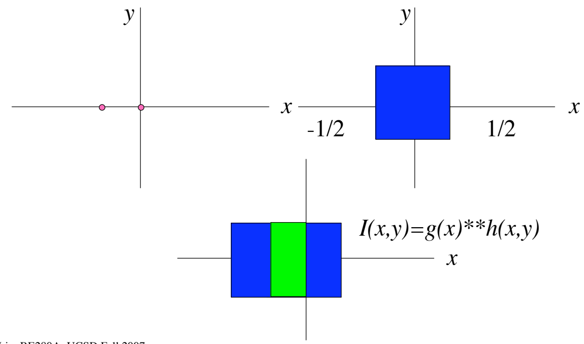


1D Convolution Examples

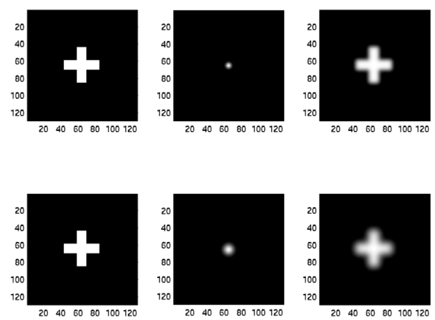


2D Convolution Example

$g(x)= \delta(x+1/2,y) + \delta(x,y)$ $h(x)=rect(x,y)$



2D Convolution Example



Pinhole Magnification Example

$$I(x_2,y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\xi,\eta)h(x_2,y_2;\xi,\eta)d\xi d\eta$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\xi,\eta)\delta(x_2 - m\xi, y_2 - m\eta)d\xi d\eta$$

after substituting $\xi' = m\xi$ and $\eta' = m\eta$, we obtain

$$= \frac{1}{m^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\xi'/m,\eta'/m)\delta(x_2 - \xi', y_2 - \eta')d\xi' d\eta'$$
$$= \frac{1}{m^2} s(x_2/m, y_2/m) ** \delta(x_2,y_2)$$
$$= \frac{1}{m^2} s(x_2/m, y_2/m)$$

X-Ray Image Equation

$$I(x_2,y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\xi,\eta)h(x_2,y_2;\xi,\eta)d\xi d\eta$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\xi,\eta)t\left(\frac{x_2 - m\xi}{M},\frac{y_2 - m\eta}{M}\right)d\xi d\eta$$

after substituting $\xi' = m\xi$ and $\eta' = m\eta$, we obtain

$$= \frac{1}{m^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\xi'/m,\eta'/m)t\left(\frac{x_2 - \xi'}{M},\frac{y_2 - \eta'}{M}\right)d\xi' d\eta'$$
$$= \frac{1}{m^2} s(x_2/m, y_2/m) ** t(x_2/M, y_2/M)$$

Note: we have ignored obliquity factors etc.

Summary

1. The response to a linear system can be characterized by a spatially varying impulse response and the application of the superposition integral.
2. A shift invariant linear system can be characterized by its impulse response and the application of a convolution integral.