Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2007
CT/Fourier Lecture 4

Topics

- Sampling Requirements in CT
- Sampling Theory
- Aliasing

CT Sampling Requirements

What should the size of the detectors be?

How many detectors do we need?

How many views do we need?

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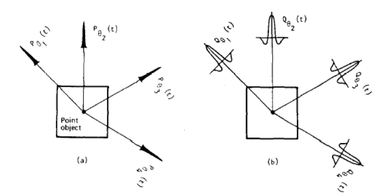
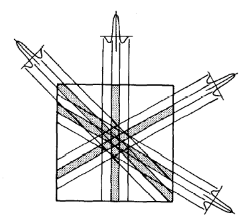
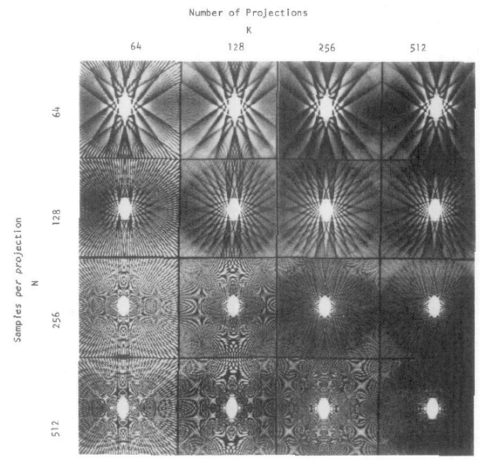
View Aliasing





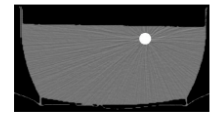
Kak and Slaney



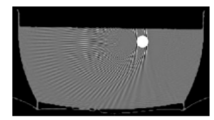
Kak and Slaney

Artifacts

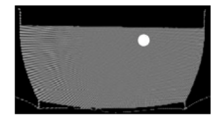
Object



Effect of Noise



Aliasing due to insufficient number of detectors



Aliasing due to insufficient number of views

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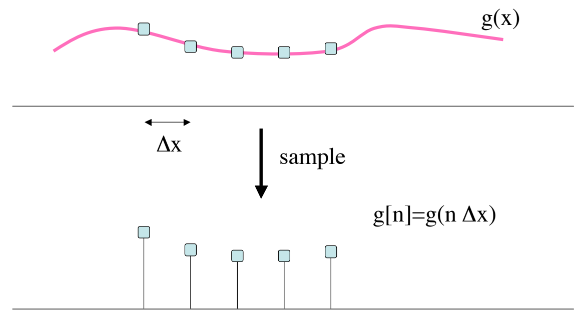
Analog vs. Digital

The Analog World:
Continuous time/space, continuous valued signals or images, e.g. vinyl records, photographs, x-ray films.

The Digital World:
Discrete time/space, discrete-valued signals or images, e.g. CD-Roms, DVDs, digital photos, digital x-rays, CT, MRI, ultrasound.

The Process of Sampling



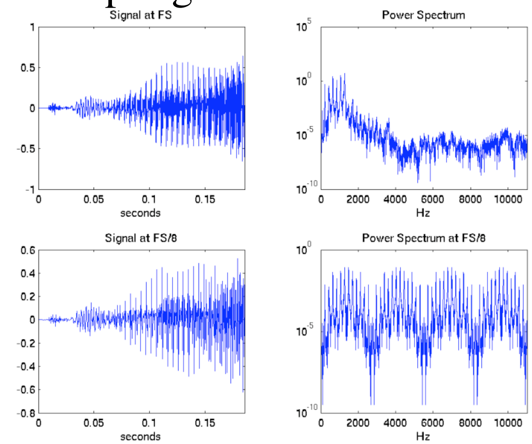
Questions

How finely do we need to sample?

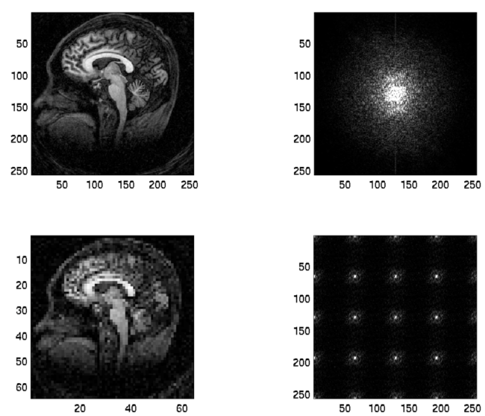
What happens if we don't sample finely enough?

Can we reconstruct the original signal or image from its samples?

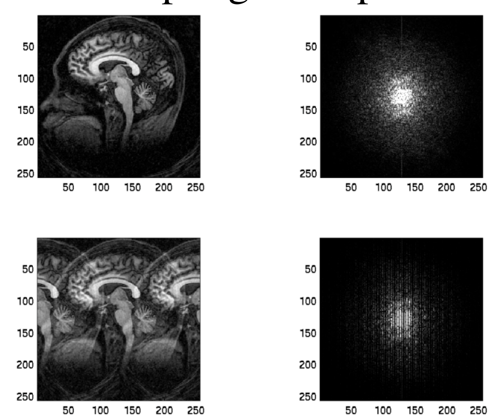
Sampling in the Time Domain



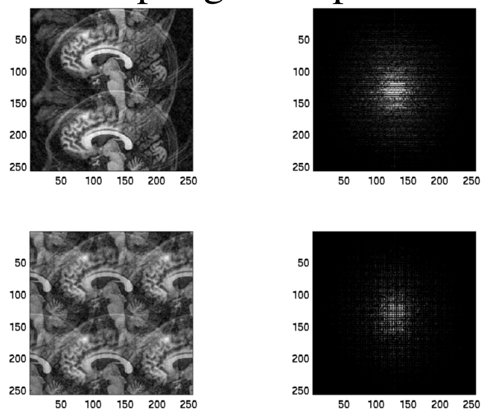
Sampling in Image Space



Sampling in k-space



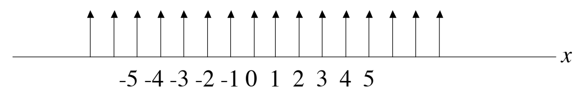
Sampling in k-space



1

Comb Function

$$comb(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$



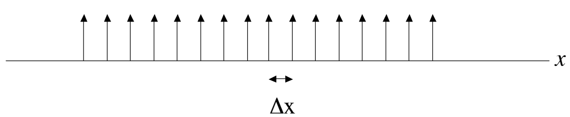
Other names: Impulse train, bed of nails, shah function.

Scaled Comb Function

$$comb\left(\frac{x}{\Delta x}\right) = \sum_{n=-\infty}^{\infty} \delta\left(\frac{x}{\Delta x} - n\right)$$

$$= \sum_{n=-\infty}^{\infty} \delta\left(\frac{x - n\Delta x}{\Delta x}\right)$$

$$= \Delta x \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x)$$



1D spatial sampling

$$g_S(x) = g(x)\frac{1}{\Delta x} comb\left(\frac{x}{\Delta x}\right)$$

$$= g(x) \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x)$$

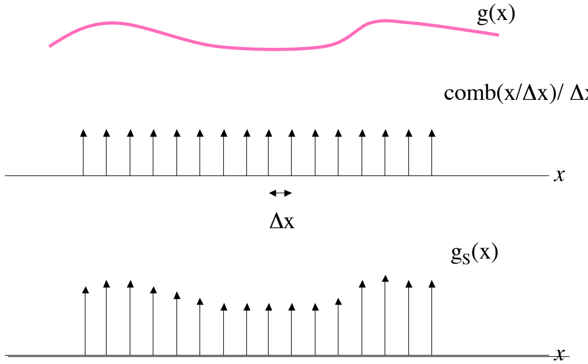
$$= \sum_{n=-\infty}^{\infty} g(n\Delta x)\delta(x - n\Delta x)$$

Recall the sifting property $\int_{-\infty}^{\infty} g(x)\delta(x-a) = g(a)$

But we can also write $\int_{-\infty}^{\infty} g(a)\delta(x-a) = g(a)\int_{-\infty}^{\infty} \delta(x-a) = g(a)$

So, $g(x)\delta(x-a) = g(a)\delta(x-a)$

1D spatial sampling



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Fourier Transform of comb(x)

$$F\left[comb(x)\right] = comb(k_x) = \sum_{n=-\infty}^{\infty} \delta(k_x - n)$$

$$F\left[\frac{1}{\Delta x} comb(\frac{x}{\Delta x})\right] = \frac{1}{\Delta x} \Delta x comb(k_x \Delta x) = \sum_{n=-\infty}^{\infty} \delta(k_x \Delta x - n) = \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} \delta(k_x - \frac{n}{\Delta x})$$

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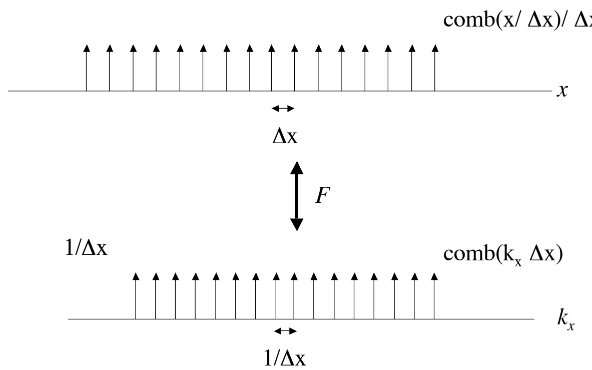
Fourier Transform of comb(x/ Δx)



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Fourier Transform of $g_S(x)$

$$\begin{aligned}
F\left[g_S(x)\right] &= F\left[g(x)\frac{1}{\Delta x} comb\left(\frac{x}{\Delta x}\right)\right] \\
&= G(k_x) * F\left[\frac{1}{\Delta x} comb\left(\frac{x}{\Delta x}\right)\right] \\
&= G(k_x) * \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} \delta\left(k_x - \frac{n}{\Delta x}\right) \\
&= \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} G(k_x) * \delta\left(k_x - \frac{n}{\Delta x}\right) \\
&= \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} G\left(k_x - \frac{n}{\Delta x}\right)
\end{aligned}$$

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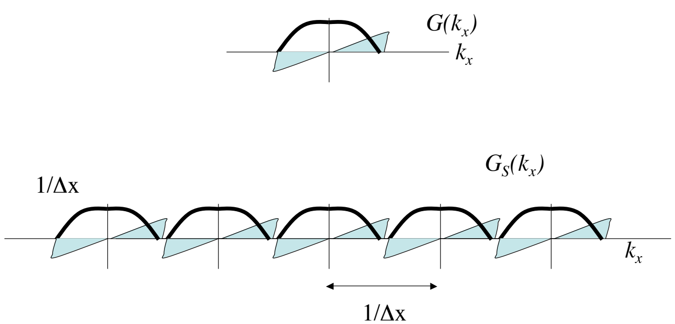
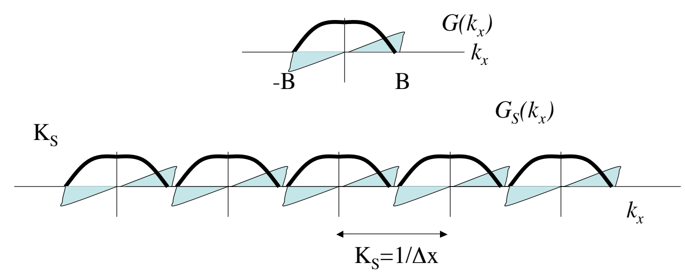
Fourier Transform of $g_S(x)$



Nyquist Condition



To avoid overlap, we require that $1/\Delta x>2B$ or
$K_S > 2B$ where $K_S=1/\Delta x$ is the sampling frequency

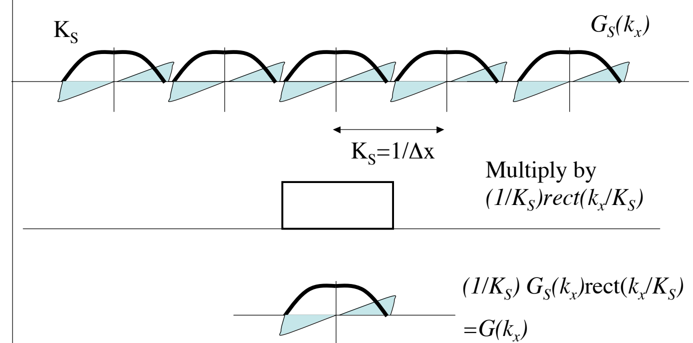
Example

Assume that the highest spatial frequency in an object is $B = 2\ \text{cm}^{-1}$.
Thus, smallest spatial period is 0.5 cm.

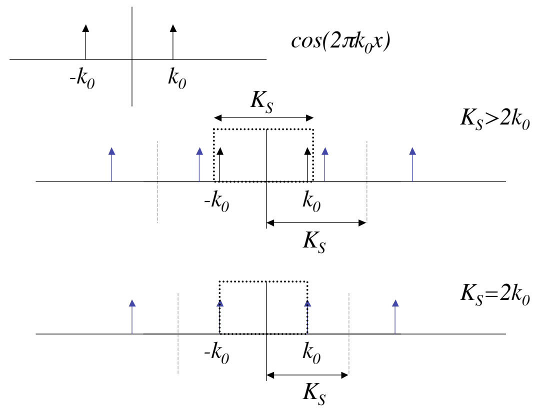
Nyquist theorem says we need to sample with
$\Delta x < 1/2B = 0.25$ cm

This corresponds to 2 samples per spatial period.

Reconstruction from Samples



Example Cosine Reconstruction



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Reconstruction from Samples

If the Nyquist condition is met, then

$$\hat{G}_S(k_x) = \frac{1}{K_S} G_S(k_x) rect(k_x / K_S) = G(k_x)$$

And the signal can be reconstructed by convolving the sample with a sinc function

$$\begin{aligned}\hat{g}_S(x) &= g_S(x) * \mathrm{sinc}(K_s x) \\ &= \left(\sum_{n=-\infty}^{\infty} g(n\Delta X)\delta(x - n\Delta X)\right) * \mathrm{sinc}(K_s x) \\ &= \sum_{n=-\infty}^{\infty} g(n\Delta X)\,\mathrm{sinc}(K_s(x - n\Delta x))\end{aligned}$$

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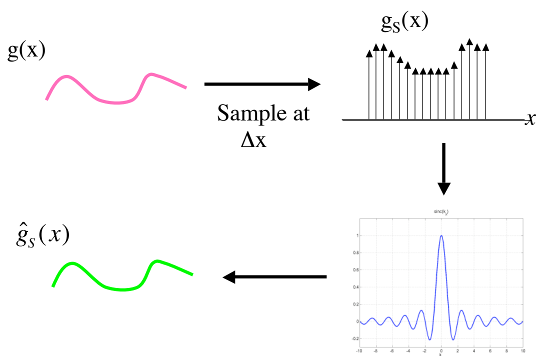
Reconstruction from Samples



$$\mathrm{sinc}(K_s x) = \mathrm{sinc}(x / \Delta x)$$

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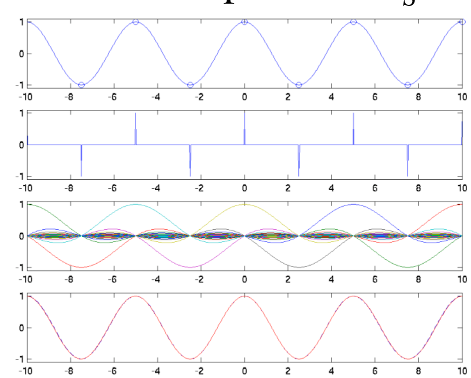
Cosine Example with $K_S=2k_0$



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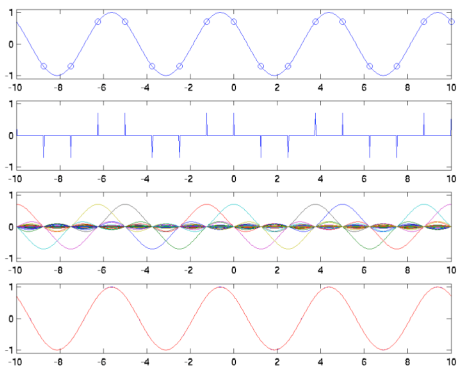
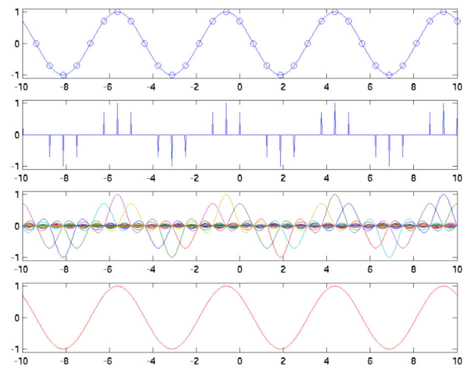
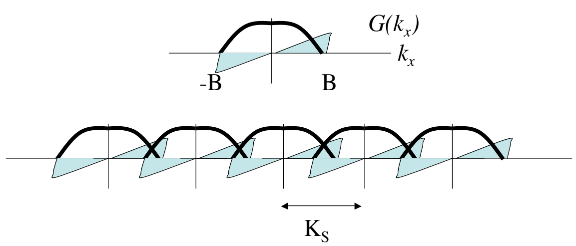
Example with $K_s=4k_0$



Example with $K_s=8k_0$

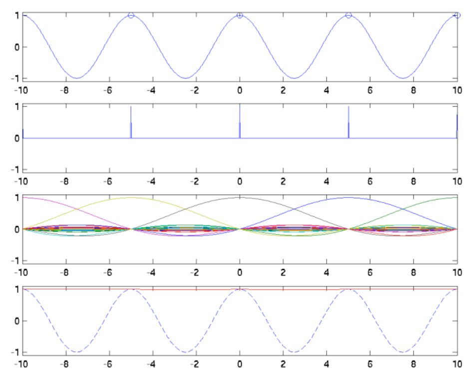


Aliasing

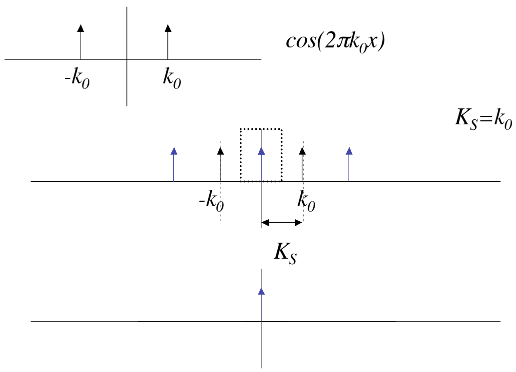


Aliasing occurs when the Nyquist condition is not satisfied.
This occurs for $K_S \leq 2B$

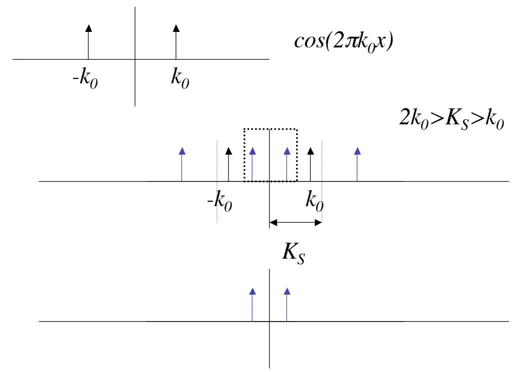
Aliasing Example



Aliasing Example



Aliasing Example



Example

1. C onsider the function $g(x) = \cos^2(2\pi k_0 x)$. Sketch this function. You sample this signal in the spatial domain with a sampling rate $K_S = 1/\Delta x$ (e.g. samples spaced at intervals of Δx). What is the minimum sampling rate that you can use without aliasing? Give an intuitive explanation for your answer.

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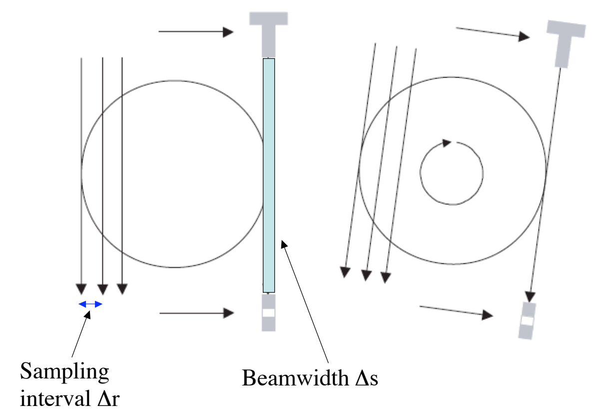
Detector Sampling Requirements



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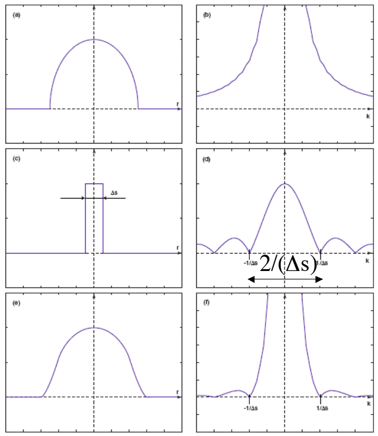
Smoothing of Projection

Projection

Beam Width

Smoothed Projection



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Smoothing of Projection

$$g_s(l,\theta) = rect(l/\Delta s) * g(l,\theta)$$

$$G_s(k_x,\theta) = \Delta s \operatorname{sinc}(k_x \Delta s) G(k_x,\theta)$$

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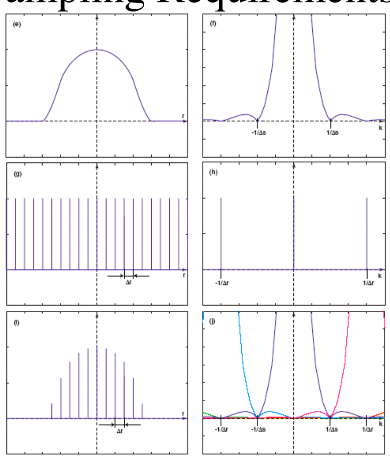
Sampling Requirements

Smoothed Projection

Detectors $\Delta r \leq \Delta s/2$

Sampled Smooth Projection



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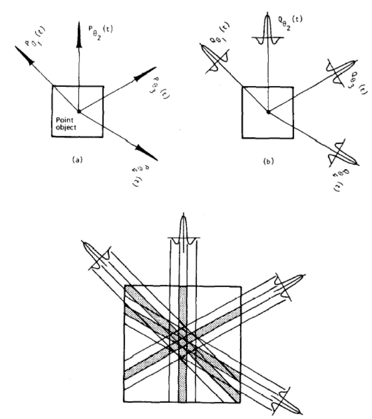
View Aliasing



Kak and Slaney

View Sampling Requirements

View Sampling -- how many views?

Basic idea is that to make the maximum angular sampling the same as the projection sampling.

$$\frac{\pi FOV}{N_{views}} = \Delta r$$

$$N_{views,360} = \frac{\pi FOV}{\Delta r} = \pi N_{proj} \quad \text{(for 360 degrees)}$$

$$N_{views,180} = \frac{\pi N_{proj}}{2} \quad \text{(for 180 degrees)}$$

Suetens 2002

Example

beamwidth Δs = 1 mm

Field of View (FOV) = 50 cm

Δr = Δs/2 = 0.5 mm

500 mm/ 0.5 mm = N = 1000 detector samples

π * N = 3146 views per 360 degrees

$\approx$ 1500 views per 180 degrees

CT "Rule of Thumb"

$$N_{view} = N_{\det ectors} = N_{pixels}$$

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