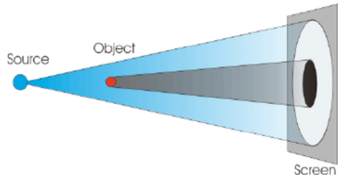
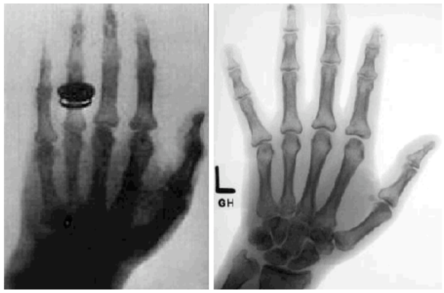
Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2008
X-Rays Lecture 1





EM spectrum

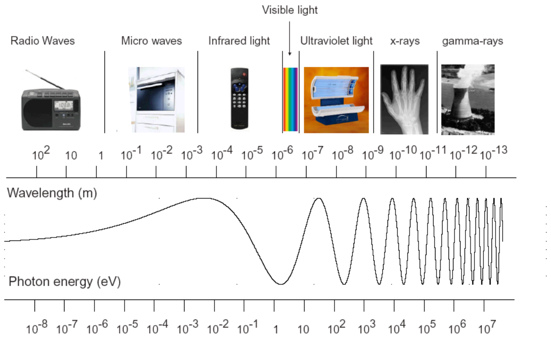
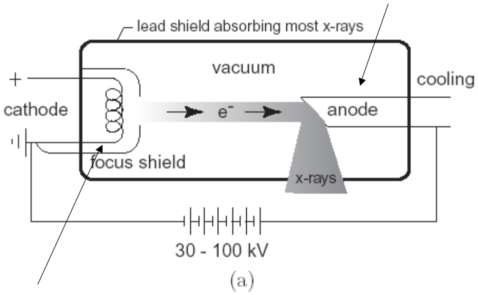


Figure 4.1: *The electromagnetic spectrum.*

Suetens 2002

X-Ray Tube

Usually tungsten is used for anode
Molybdenum for mammography



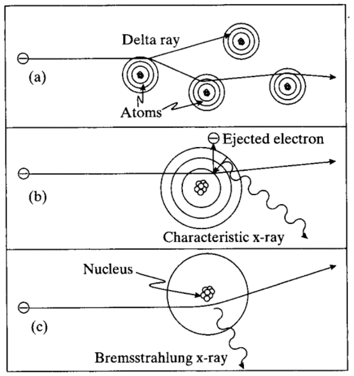
Tungsten filament heated to about 2200 C leading to thermionic emission of electrons.

Suetens 2002

X-Ray Production

Collisional transfers

Radiative transfers



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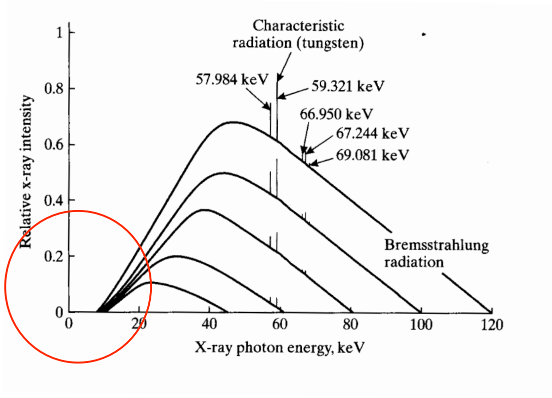
X-Ray Spectrum

Lower energy photons are absorbed by anode, tube, and other filters



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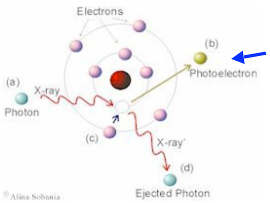
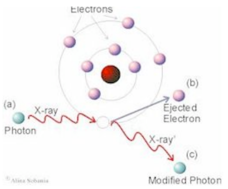
Interaction with Matter

Typical energy range for diagnostic x-rays is below 200 keV. The two most important types of interaction are photoelectric absorption and Compton scattering.



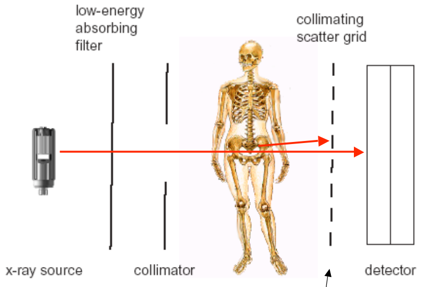
Photoelectric effect dominates at low x-ray energies and high atomic numbers. $E_{e^-} = \hbar\nu - E_B$



Compton scattering dominates at high x-ray energies and low atomic numbers, not much contrast

http://www.eee.ntu.ac.uk/research/vision/asobania

X-Ray Imaging Chain



Reduces effects of Compton scattering

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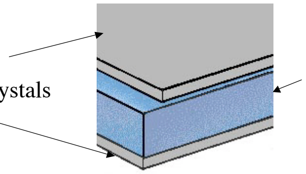
X-ray film

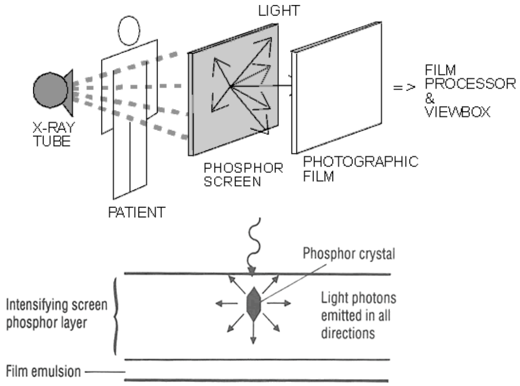


Emulsion with
silver halide crystals
Each layer
~ 10 µm

Flexible base
~ 150 µm

Silver halide crystals absorb optical energy. After development, crystals that have absorbed enough energy are converted to metallic silver and look dark on the screen. Thus, parts in the object that attenuate the x-rays will look brighter.

Intensifying Screen



http://learntech.uwe.ac.uk/radiography/RScience/imaging_principles_d/diagimage11.htm
http://www.sunnybrook.utoronto.ca:8080/~selenium/xray.html#Film

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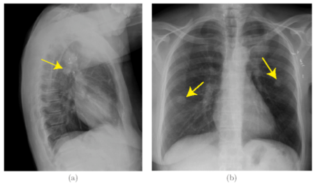
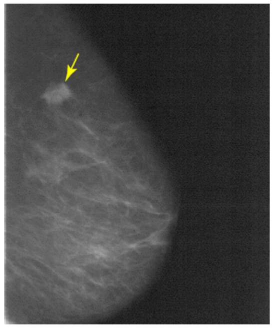
X-Ray Examples





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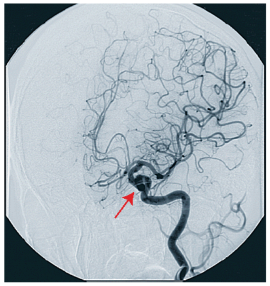
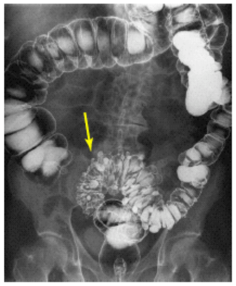
X-Ray w/ Contrast Agents



Angiogram using an iodine-based contrast agent.
K-edge of iodine is 33.2 keV



Barium Sulfate
K-edge of Barium is 37.4 keV

Suetens 2002

Intensity

$I = E\phi$

Energy

Photon flux rate

Number of photons

$$\phi = \frac{N}{A\Delta t}$$

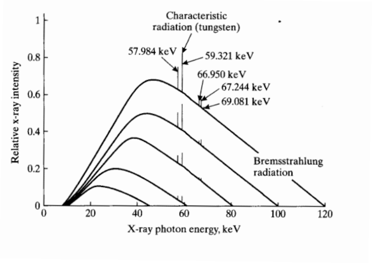
Unit Area

Unit Time

Intensity

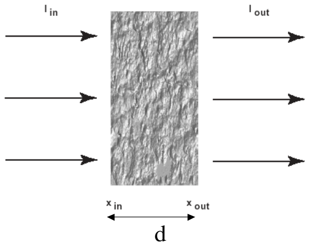
$$\phi = \int_0^\infty S(E')dE'$$

X-ray spectrum



$$I = \int_0^\infty S(E')E'dE'$$

Attenuation



For single-energy x-rays passing through a homogenous object:

$$I_{out} = I_{in}\exp(-\mu d)$$

Linear attenuation coefficient

Attenuation

$n = \mu N \Delta x$ photons lost per unit length

$\mu = \frac{n/N}{\Delta x}$ fraction of photons lost per unit length

$$\Delta N = -n \longrightarrow \frac{dN}{dx} = -\mu N \longrightarrow N(x) = N_0 e^{-\mu x}$$

For mono-energetic case, intensity is

$$I(\Delta x) = I_0 e^{-\mu \Delta x}$$

Attenuation

Inhomogeneous Slab

$$\frac{dN}{dx} = -\mu(x)N \longrightarrow N(x) = N_0 \exp\left(-\int_0^x \mu(x')dx'\right)$$

$$I(x) = I_0 \exp\left(-\int_0^x \mu(x')dx'\right)$$

Attenuation depends on energy, so also need to integrate over energies

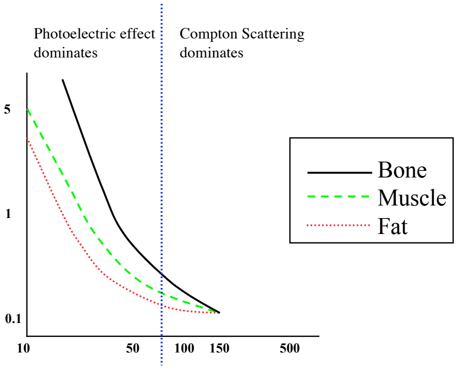
$$I(x) = \int_0^\infty S_0(E')E' \exp\left(-\int_0^x \mu(x';E')dx'\right)dE'$$

Attenuation

More Attenuation

Attenuation Coefficient

Less Attenuation



Photon Energy (keV)

Adapted from www.cis.rit.edu/class/simg215/xrays.ppt

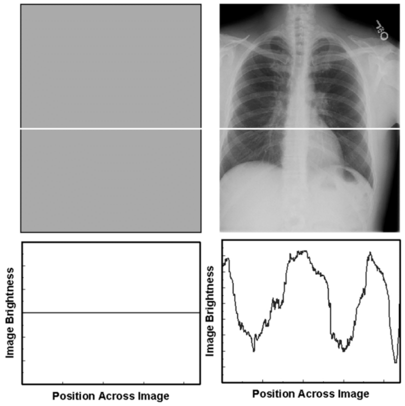
Half Value Layer

X-ray energy (keV)	HVL, muscle (cm)	HVL Bone (cm)
30	1.8	0.4
50	3.0	1.2
100	3.9	2.3
150	4.5	2.8

In chest radiography, about 90% of x-rays are absorbed by body. Average energy from a tungsten source is 68 keV. However, many lower energy beams are absorbed by tissue, so average energy is higher. This is referred to as beam-hardening, and reduces the contrast.

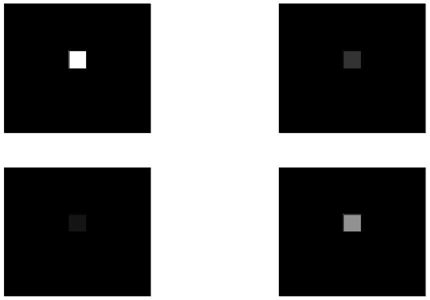
Values from Webb 2003

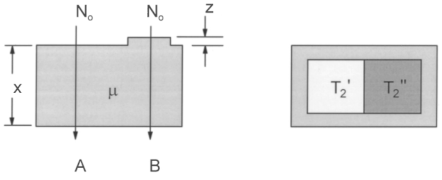
Contrast



Bushberg et al 2001

Contrast





(A) X-ray Imaging

(B) MR Imaging

Bushberg et al 2001

$$A = N_0 \exp(-\mu x)$$

$$B = N_0 \exp(-\mu(x+z))$$

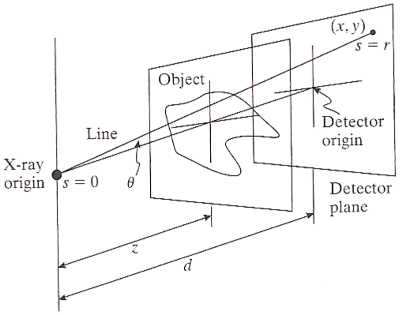
Subject Contrast

$$C_S = \frac{A-B}{A}$$

$$= \frac{N_0 \exp(-\mu x) - N_0 \exp(-\mu(x+z))}{N_0 \exp(-\mu x)}$$

$$= 1 - \exp(-\mu z)$$

X-Ray Imaging Geometry



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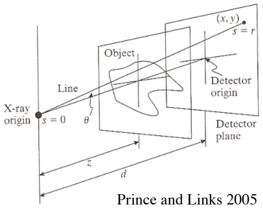
Inverse Square Law

Inverse Square Law

$$I_0 = \frac{I_S}{4\pi d^2}$$

$$I_d(x,y) = \frac{I_S}{4\pi r^2} \quad \text{where } r^2 = x^2 + y^2 + d^2$$

$$= \frac{I_0 d^2}{r^2} = I_0 \cos^2 \theta$$



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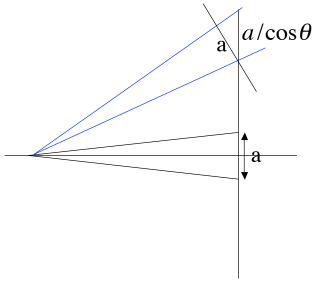
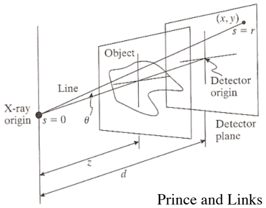
Obliquity Factor

Obliquity Factor

$I_d(x,y) = I_0 \cos\theta$





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X-Ray Imaging Geometry

Beam Divergence and Flat Panel

$I_r = I_0 \cos^3\theta$

Example : Chest x - ray at 2 yards with 14x17 inch film.

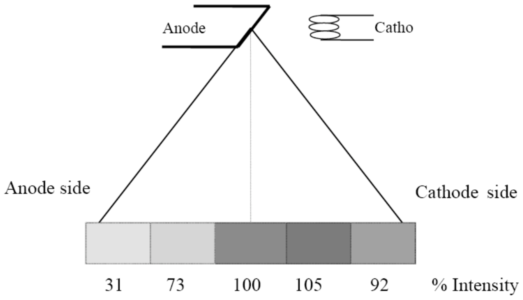
Question : What is the smallest ratio I_r/I_0 across the film?

$$r_d = \sqrt{7^2 + 8.5^2} = 11$$

$$\cos\theta = \frac{d}{\sqrt{r_d^2 + d^2}} = 0.989$$

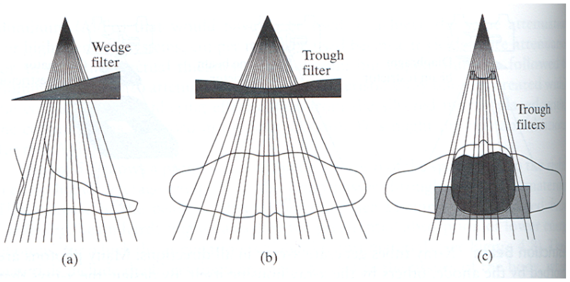
$$\frac{I_r}{I_0} = \cos^3\theta = 0.966$$

Anode Heel Effect



http://www.animalinsides.com/radphys/chapters/Lect2.pdf

Compensation Filters



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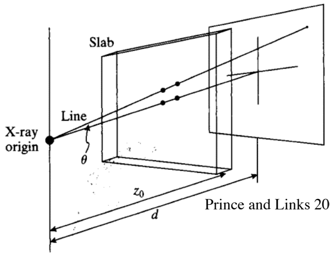
Path Length



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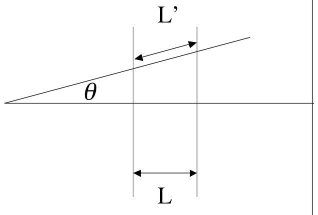
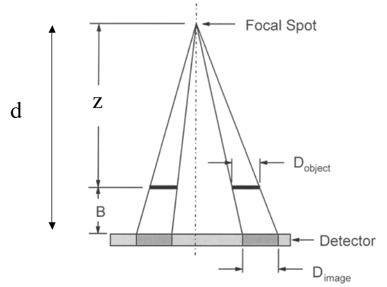


$$L' = L/\cos\theta$$

$$I_d(x,y) = I_0\cos^3\theta\exp(-\mu L/\cos\theta)$$

Magnification of Object

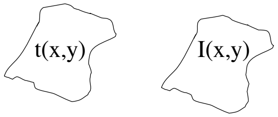


$$M(z) = \frac{d}{z}$$

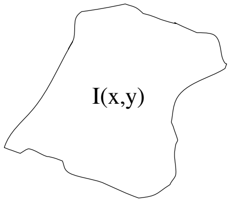
$$= \frac{\text{Source to Image Distance (SID)}}{\text{Source to Object Distance (SOD)}}$$

Bushberg et al 2001

Magnification of Object



M = 1: I(x,y) = t(x,y)



M = 2: I(x,y) = t(x/2,y/2)

In general, I(x,y) = t(x/M(z),y/M(z))

X-Ray Imaging Equation

At z = d there is no magnification, so

$$I_d(x,y) = I_0\cos^3\theta\cdot\exp\left(-\int_{L_{x,y}}\mu(s)ds\Big/\cos\theta\right)$$

$$= I_0\cos^3\theta\cdot t_d(x,y)$$

where $t_z(x,y)$ is the transmittivity of the object at distance z

In general, with magnification

$$I_d(x,y) = I_0\cos^3\theta\cdot t_z(x/M(z),y/M(z))$$

Prince and Links 2005

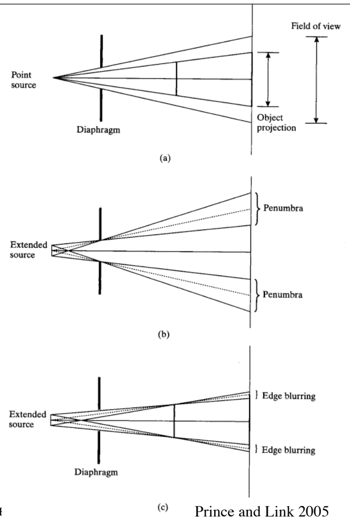
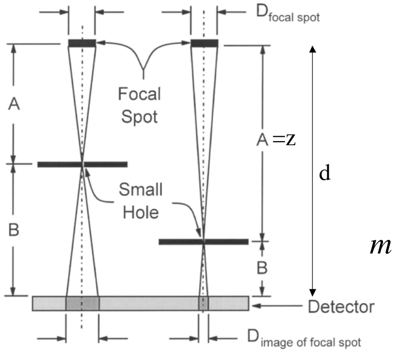


Figure 5.18
Effects of extended source. (a) Ideal field of view and object projection (with magnification). (b) Penumbra at edges of field of view due to extended source. (c) Blurred object edges due to extended source.

Prince and Link 2005

Source magnification



Bushberg et al 2001

$$\frac{D_{image}}{D_{focal}} = \frac{d-z}{z}$$

$$m(z) = -\frac{d-z}{z} = -\frac{B}{A}$$
$$= 1 - M(z)$$

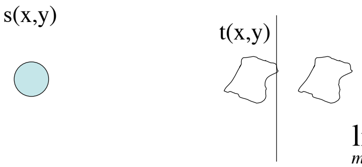
Image of a point object

$$I_d(x,y) = ks(x/m, y/m)$$

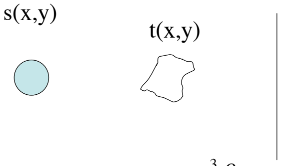
$$\int\int ks\left(x/m(z), y/m(z)\right)dxdy = \text{ constant}$$
$$\Rightarrow k = \frac{1}{m^2(z)}$$

$$I_d(x,y) = \lim_{m\to 0} \frac{s(x/m, y/m)}{m^2}$$
$$= \delta(x,y)$$

Image of arbitrary object



$$\lim_{m\to 0} I_d(x,y) = t(x,y)$$

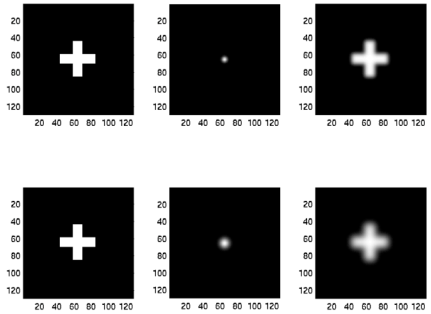


m=1

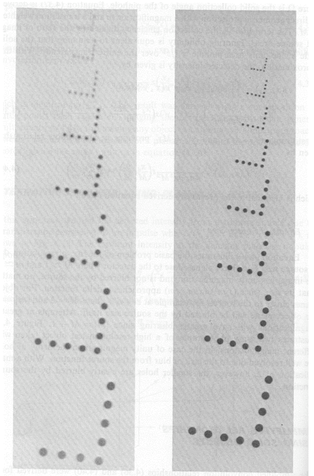
$$I_d(x,y) = ???$$

$$I_d(x,y) = \frac{\cos^3\theta}{4\pi d^2 m^2} s(x/m, y/m) ** t(x/M, y/M)$$

Convolution



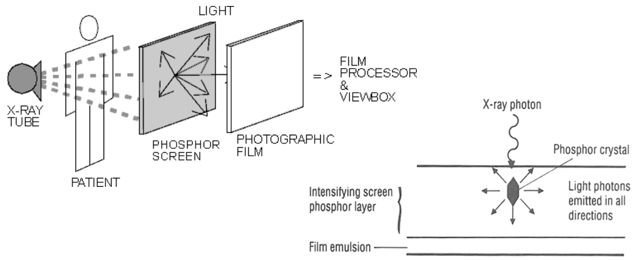
M=2
m=-1



M=1
m=0

Macovski 1983

Film-screen blurring



$$I_d(x,y) = \frac{\cos^3\theta}{4\pi d^2 m^2} s(x/m, y/m) ** t(x/M, y/M) ** h(x,y)$$

http://learntech.uwe.ac.uk/radiography/RScience/imaging_principles_d/diagimage11.htm
http://www.sunnybrook.utoronto.ca:8080/~selenium/xray.html#Film

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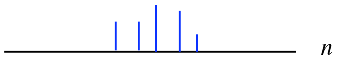
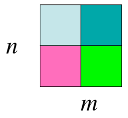
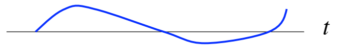
Signals and Images

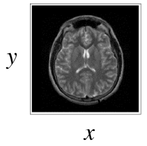
Discrete-time/space signałimage: continuous valued function with a discrete time/space index, denoted as *s[n]* for 1D, *s[m,n]* for 2D , etc.





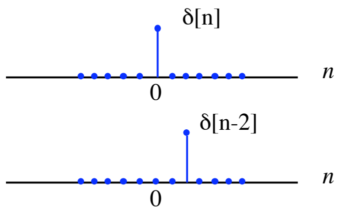
Continuous-time/space signałimage: continuous valued function with a continuous time/space index, denoted as *s(t)* or *s(x)* for 1D, *s(x,y)* for 2D, etc.





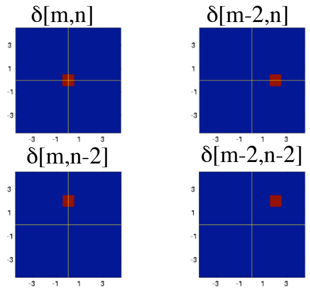
Kronecker Delta Function

$$\delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases}$$



Kronecker Delta Function

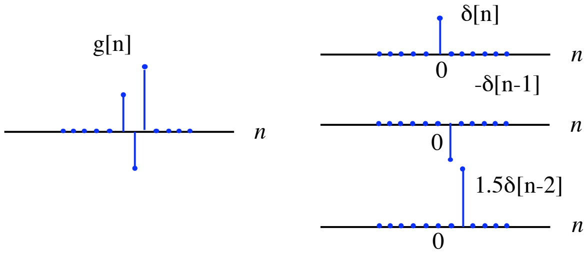
$$\delta[m,n] = \begin{cases} 1 & \text{for } m = 0, n = 0 \\ 0 & \text{otherwise} \end{cases}$$



Discrete Signal Expansion

$$g[n] = \sum_{k=-\infty}^{\infty} g[k]\delta[n-k]$$

$$g[m,n] = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} g[k,l]\delta[m-k,n-l]$$



2D Signal

$$\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array} = \begin{array}{|c|c|}\hline a & 0 \\ \hline 0 & 0 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 0 & b \\ \hline 0 & 0 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 0 & 0 \\ \hline c & 0 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & d \\ \hline\end{array}$$

Image Decomposition

$$\begin{array}{|c|c|}\hline c & d \\ \hline a & b \\ \hline \end{array} = c\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 0 \\ \hline \end{array} + d\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 0 & 0 \\ \hline \end{array} + a\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & 0 \\ \hline \end{array} + b\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & 1 \\ \hline \end{array}$$

$$g[m,n] = a\delta[m,n] + b\delta[m,n-1] + c\delta[m-1,n] + d\delta[m-1,n-1]$$

$$= \sum_{k=0}^{1}\sum_{l=0}^{1} g[k,l]\delta[m-k,n-l]$$

Dirac Delta Function

Notation :

$\delta(x)$ - 1D Dirac Delta Function

$\delta(x,y)$ or ${}^2\delta(x,y)$ - 2D Dirac Delta Function

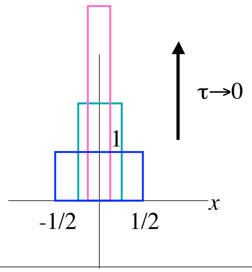
$\delta(x,y,z)$ or ${}^3\delta(x,y,z)$ - 3D Dirac Delta Function

$\delta(\vec{r})$ - N Dimensional Dirac Delta Function

1D Dirac Delta Function

$\delta(x) \;=\; 0$ when $x \neq 0$ and $\int_{-\infty}^{\infty}\delta(x)dx = 1$

Can interpret the integral as a limit of the integral of an ordinary function that is shrinking in width and growing in height, while maintaining a constant area. For example, we can use a shrinking rectangle function such that $\int_{-\infty}^{\infty}\delta(x)dx = \lim_{\tau\to 0}\int_{-\infty}^{\infty}\tau^{-1}\Pi(x/\tau)dx$.



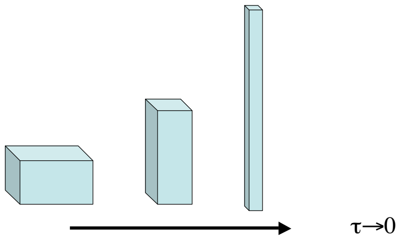
2D Dirac Delta Function

$\delta(x,y) = 0$ when $x^2 + y^2 \neq 0$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x,y)dxdy = 1$

where we can consider the limit of the integral of an ordinary 2D function that is shrinking in width but increasing in height while maintaining constant area.

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x,y)dxdy = \lim_{\tau\to 0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tau^{-2}\Pi(x/\tau, y/\tau)dxdy.$

Useful fact : $\delta(x,y) = \delta(x)\delta(y)$

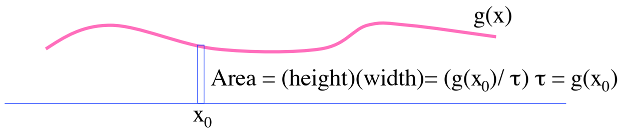


Generalized Functions

Dirac delta functions are not ordinary functions that are defined by their value at each point. Instead, they are generalized functions that are defined by what they do underneath an integral.

The most important property of the Dirac delta is the sifting property $\int_{-\infty}^{\infty}\delta(x-x_0)g(x)dx = g(x_0)$ where $g(x)$ is a smooth function. This sifting property can be understood by considering the limiting case

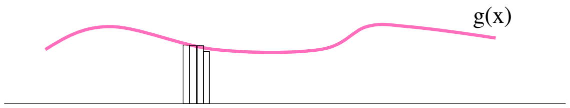
$$\lim_{\tau\to 0}\int_{-\infty}^{\infty}\tau^{-1}\Pi(x/\tau)g(x)dx = g(x_0)$$



Representation of 1D Function

From the sifting property, we can write a 1D function as $g(x)=\int_{-\infty}^{\infty}g(\xi)\delta(x-\xi)d\xi$. To gain intuition, consider the approximation

$$g(x)\approx\sum_{n=-\infty}^{\infty}g(n\Delta x)\frac{1}{\Delta x}\Pi\left(\frac{x-n\Delta x}{\Delta x}\right)\Delta x.$$



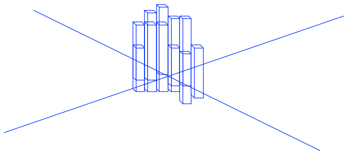
Representation of 2D Function

Similarly, we can write a 2D function as

$$g(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(\xi,\eta)\delta(x-\xi,y-\eta)d\xi d\eta.$$

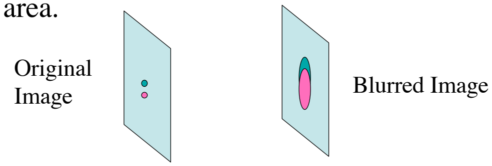
To gain intuition, consider the approximation

$$g(x,y)\approx\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}g(n\Delta x,m\Delta y)\frac{1}{\Delta x}\Pi\left(\frac{x-n\Delta x}{\Delta x}\right)\frac{1}{\Delta y}\Pi\left(\frac{y-m\Delta y}{\Delta y}\right)\Delta x\Delta y.$$

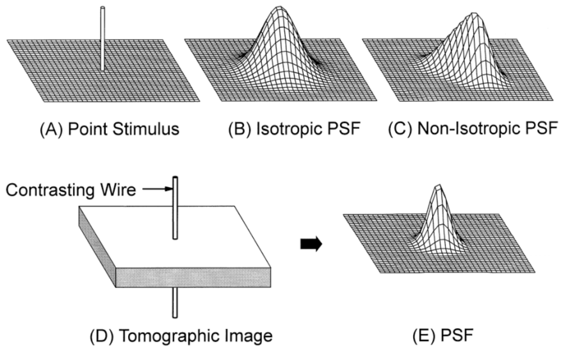


Impulse Response

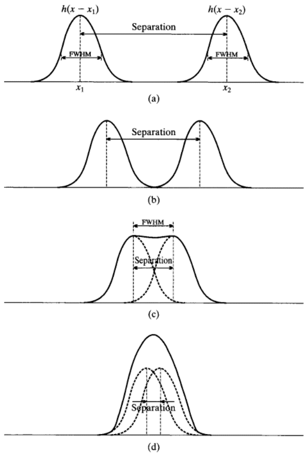
Intuition: the impulse response is the response of a system to an input of infinitesimal width and unit area.



Since any input can be thought of as the weighted sum of impulses, a linear system is characterized by its impulse response(s).



Bushberg et al 2001



Full Width Half Maximum (FWHM) is a measure of resolution.

Figure 3.6 An example of the effect of system resolution on the ability to differentiate two points. The FWHM equals the minimum distance that the two points must be separated in order to be distinguishable.

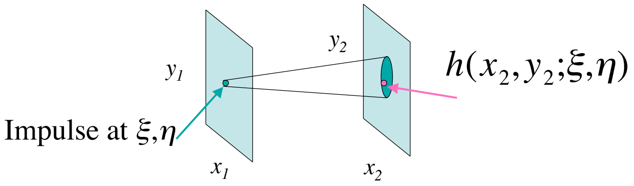
Prince and Link 2005

Impulse Response

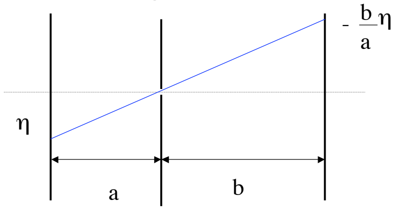
The impulse response characterizes the response of a system over all space to a Dirac delta impulse function at a certain location.

$h(x_2;\xi) = L\left[\delta(x_1-\xi)\right]$ 1D Impulse Response

$h(x_2,y_2;\xi,\eta) = L\left[\delta(x_1-\xi,y_1-\eta)\right]$ 2D Impulse Response



Pinhole Magnification Example



In this example, an impulse at (ξ,η) will yield an impulse at $(m\xi,m\eta)$ where $m=-b/a$.

Thus, $h(x_2,y_2;\xi,\eta) = L\left[\delta(x_1-\xi,y_1-\eta)\right] = \delta(x_2-m\xi,y_2-m\eta)$.