Bioengineering 280A
Principles of Biomedical Imaging

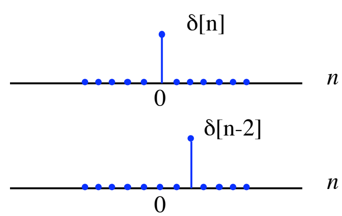
Fall Quarter 2008
X-Rays Lecture 2

Topics

- Review of Signal Expansions
- Linearity
- Superposition
- Convolution

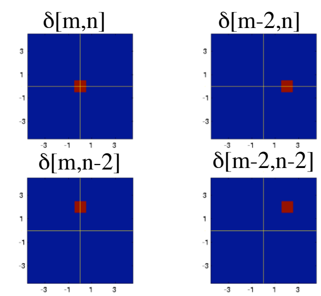
Kronecker Delta Function

$$\delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & otherwise \end{cases}$$



Kronecker Delta Function

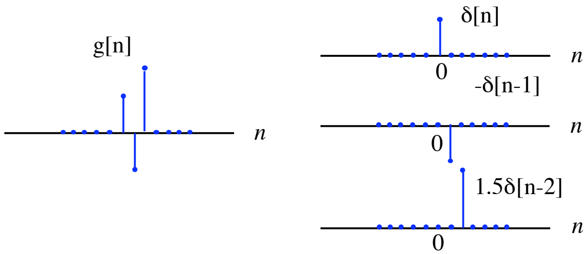
$$\delta[m,n] = \begin{cases} 1 & \text{for } m = 0, n = 0 \\ 0 & otherwise \end{cases}$$



Discrete Signal Expansion

$$g[n]=\sum_{k=-\infty}^{\infty} g[k]\delta[n-k]$$

$$g[m,n]=\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} g[k,l]\delta[m-k,n-l]$$



2D Signal

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$$

Image Decomposition

$$\begin{bmatrix} c & d \\ a & b \end{bmatrix} = c\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$g[m,n]=a\delta[m,n]+b\delta[m,n-1]+c\delta[m-1,n]+d\delta[m-1,n-1]$$

$$=\sum_{k=0}^{1}\sum_{l=0}^{1} g[k,l]\delta[m-k,n-l]$$

Dirac Delta Function

Notation:

$\delta(x)$ - 1D Dirac Delta Function

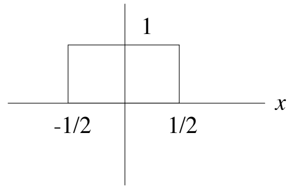
$\delta(x,y)$ or ${}^2\delta(x,y)$ - 2D Dirac Delta Function

$\delta(x,y,z)$ or ${}^3\delta(x,y,z)$ - 3D Dirac Delta Function

$\delta(\vec{r})$ - N Dimensional Dirac Delta Function

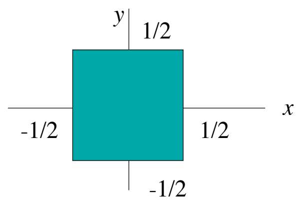
Rectangle Function

$$\Pi(x) = \begin{cases} 0 & |x| > 1/2 \\ 1 & |x| \le 1/2 \end{cases}$$



Also called rect(x)

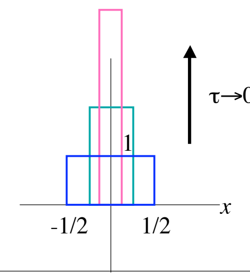
$$\Pi(x,y) = \Pi(x)\Pi(y)$$



1D Dirac Delta Function

$\delta(x) = 0$ when $x \neq 0$ and $\int_{-\infty}^{\infty} \delta(x)dx = 1$

Can interpret the integral as a limit of the integral of an ordinary function that is shrinking in width and growing in height, while maintaining a constant area. For example, we can use a shrinking rectangle function such that $\int_{-\infty}^{\infty} \delta(x)dx = \lim_{\tau \to 0} \int_{-\infty}^{\infty} \tau^{-1}\Pi(x/\tau)dx$.



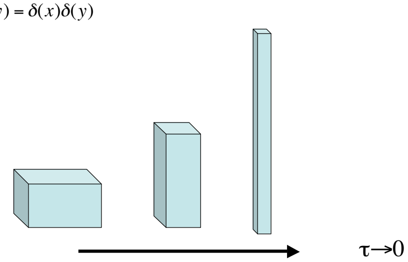
2D Dirac Delta Function

$\delta(x,y) = 0$ when $x^2 + y^2 \neq 0$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)dxdy = 1$

where we can consider the limit of the integral of an ordinary 2D function that is shrinking in width but increasing in height while maintaining constant area.

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)dxdy = \lim_{\tau \to 0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tau^{-2}\Pi(x/\tau, y/\tau)dxdy$.

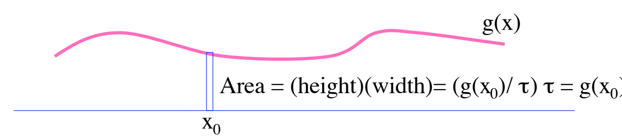
Useful fact : $\delta(x,y) = \delta(x)\delta(y)$



Generalized Functions

Dirac delta functions are not ordinary functions that are defined by their value at each point. Instead, they are generalized functions that are defined by what they do underneath an integral.

The most important property of the Dirac delta is the sifting property $\int_{-\infty}^{\infty} \delta(x - x_0)g(x)dx = g(x_0)$ where $g(x)$ is a smooth function. This sifting property can be understood by considering the limiting case $\lim_{\tau \to 0} \int_{-\infty}^{\infty} \tau^{-1}\Pi(x/\tau)g(x)dx = g(x_0)$

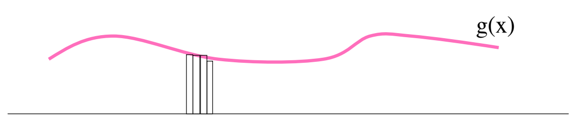


Representation of 1D Function

From the sifting property, we can write a 1D function as

$g(x) = \int_{-\infty}^{\infty} g(\xi)\delta(x-\xi)d\xi.$ To gain intuition, consider the approximation

$$g(x) \approx \sum_{n=-\infty}^{\infty} g(n\Delta x)\frac{1}{\Delta x}\Pi\left(\frac{x-n\Delta x}{\Delta x}\right)\Delta x.$$



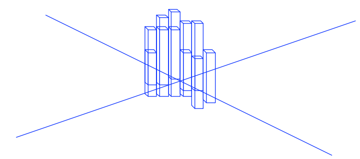
Representation of 2D Function

Similarly, we can write a 2D function as

$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x-\xi,y-\eta)d\xi d\eta.$

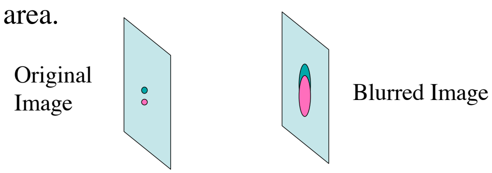
To gain intuition, consider the approximation

$$g(x,y) \approx \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g(n\Delta x, m\Delta y)\frac{1}{\Delta x}\Pi\left(\frac{x-n\Delta x}{\Delta x}\right)\frac{1}{\Delta y}\Pi\left(\frac{y-m\Delta y}{\Delta y}\right)\Delta x\Delta y.$$

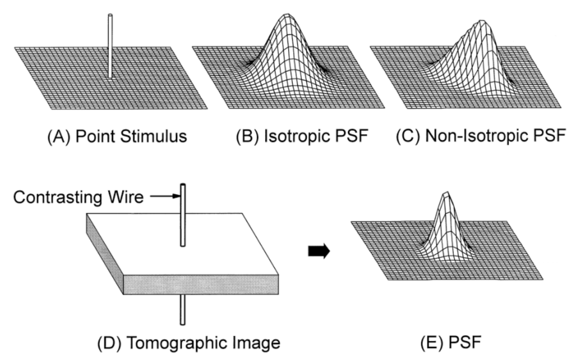


Impulse Response

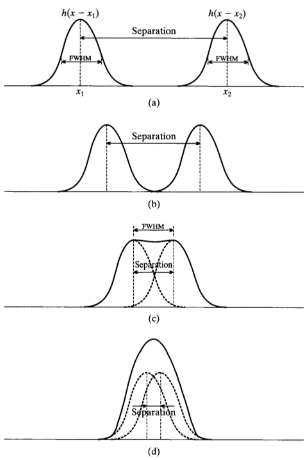
Intuition: the impulse response is the response of a system to an input of infinitesimal width and unit area.



Since any input can be thought of as the weighted sum of impulses, a linear system is characterized by its impulse response(s).



Bushberg et al 2001



Full Width Half Maximum (FWHM) is a measure of resolution.

Figure 3.6
An example of the effect of system resolution on the ability to differentiate two points. The FWHM equals the minimum distance that the two points must be separated in order to be distinguishable.

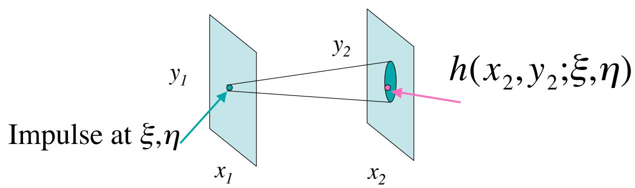
Prince and Link 2005

Impulse Response

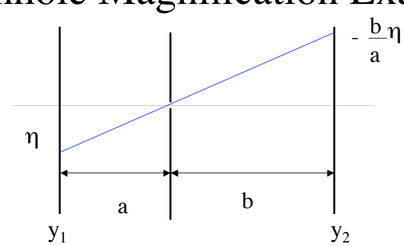
The impulse response characterizes the response of a system over all space to a Dirac delta impulse function at a certain location.

$h(x_2;\xi) = L[\delta(x_1 - \xi)]$ 1D Impulse Response

$h(x_2,y_2;\xi,\eta) = L[\delta(x_1 - \xi, y_1 - \eta)]$ 2D Impulse Response



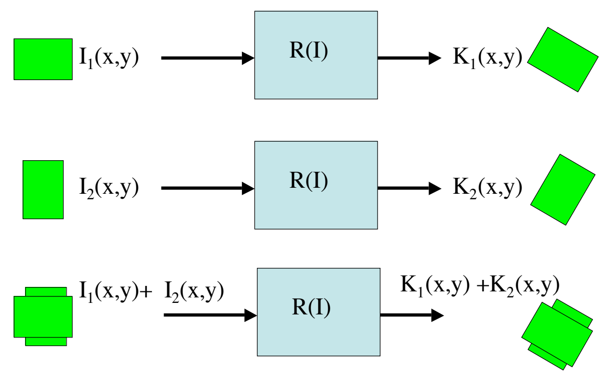
Pinhole Magnification Example



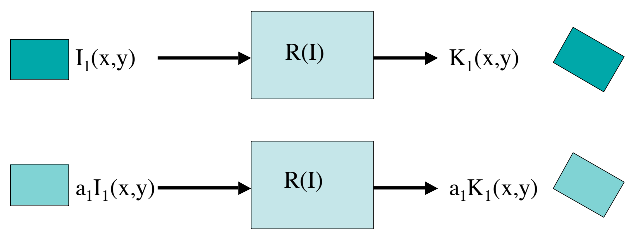
In this example, an impulse at (ξ,η) will yield an impulse at $(m\xi, m\eta)$ where $m = -b/a$.

Thus, $h(x_2,y_2;\xi,\eta) = L[\delta(x_1 - \xi, y_1 - \eta)] = \delta(x_2 - m\xi, y_2 - m\eta)$.

Linearity (Addition)



Linearity (Scaling)



Linearity

A system R is linear if for two inputs $I_1(x,y)$ and $I_2(x,y)$ with outputs

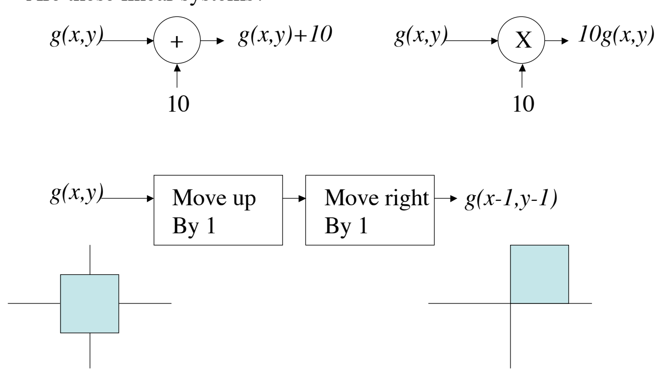
$R(I_1(x,y))=K_1(x,y)$ and $R(I_2(x,y))=K_2(x,y)$

the response to the weighted sum of inputs is the weighted sum of outputs:

$R(a_1I_1(x,y)+ a_2I_2(x,y))=a_1K_1(x,y)+ a_2K_2(x,y)$

Example

Are these linear systems?



Superposition

$$g[m] = g[0]\delta[m] + g[1]\delta[m-1] + g[2]\delta[m-2]$$

$$h[m',k] = L[\delta[m-k]]$$

$$\begin{aligned}
y[m'] &= L\left[g[m]\right] \\
&= L\left[g[0]\delta[m] + g[1]\delta[m-1] + g[2]\delta[m-2]\right] \\
&= L\left[g[0]\delta[m]\right] + L\left[g[1]\delta[m-1]\right] + L\left[g[2]\delta[m-2]\right] \\
&= g[0]L\left[\delta[m]\right] + g[1]L\left[\delta[m-1]\right] + g[2]L\left[\delta[m-2]\right] \\
&= g[0]h[m',0] + g[1]h[m',1] + g[2]h[m',2] \\
&= \sum_{k=0}^{2} g[k]h[m',k]
\end{aligned}$$

Superposition Integral

What is the response to an arbitrary function $g(x_1,y_1)$?

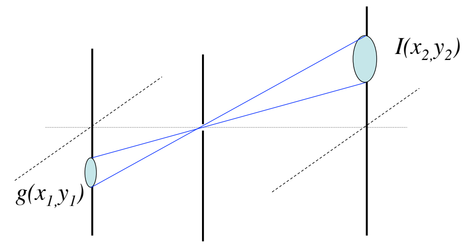
Write $g(x_1,y_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x_1-\xi,y_1-\eta)d\xi d\eta.$

The response is given by

$$\begin{aligned} I(x_2,y_2) &= L\left[g_1(x_1,y_1)\right] \\ &= L\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x_1-\xi,y_1-\eta)d\xi d\eta\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)L\left[\delta(x_1-\xi,y_1-\eta)\right]d\xi d\eta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2,y_2;\xi,\eta)d\xi d\eta \end{aligned}$$

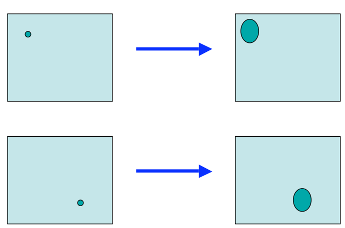
Pinhole Magnification Example

$$\begin{aligned} I(x_2,y_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2,y_2;\xi,\eta)d\xi d\eta \\ &= C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x_2-m\xi,y_2-m\eta)d\xi d\eta \end{aligned}$$

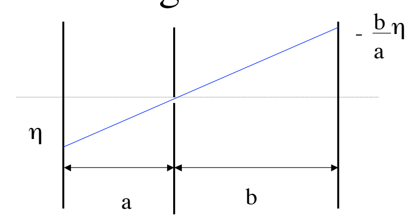


Space Invariance

If a system is space invariant, the impulse response depends only on the difference between the output coordinates and the position of the impulse and is given by $h(x_2,y_2;\xi,\eta) = h(x_2-\xi,y_2-\eta)$



Pinhole Magnification Example

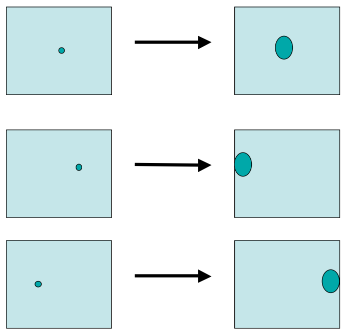


$h(x_2,y_2;\xi,\eta) = C\delta(x_2-m\xi,y_2-m\eta)$.

Is this system space invariant?

Pinhole Magnification Example

____, the pinhole system ____ space invariant.



Convolution

$$g[m] = g[0]\delta[m] + g[1]\delta[m-1] + g[2]\delta[m-2]$$

$$h[m',k] = L[\delta[m-k]] = h[m'-k]$$

$$\begin{aligned} y[m'] &= L\big[g[m]\big] \\ &= L\big[g[0]\delta[m] + g[1]\delta[m-1] + g[2]\delta[m-2]\big] \\ &= L\big[g[0]\delta[m]\big] + L\big[g[1]\delta[m-1]\big] + L\big[g[2]\delta[m-2]\big] \\ &= g[0]L\big[\delta[m]\big] + g[1]L\big[\delta[m-1]\big] + g[2]L\big[\delta[m-2]\big] \\ &= g[0]h[m'-0] + g[1]h[m'-1] + g[2]h[m'-2] \\ &= \sum_{k=0}^{2} g[k]h[m'-k] \end{aligned}$$

1D Convolution

$$\begin{aligned} I(x) &= \int_{-\infty}^{\infty} g(\xi)h(x;\xi)d\xi \\ &= \int_{-\infty}^{\infty} g(\xi)h(x-\xi)d\xi \\ &= g(x) * h(x) \end{aligned}$$

Useful fact:

$$\begin{aligned} g(x) * \delta(x-\Delta) &= \int_{-\infty}^{\infty} g(\xi)\delta(x-\Delta-\xi)d\xi \\ &= g(x-\Delta) \end{aligned}$$

2D Convolution

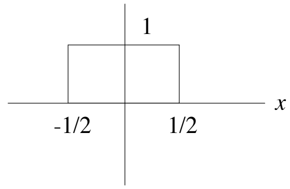
For a space invariant linear system, the superposition integral becomes a convolution integral.

$$\begin{aligned} I(x_2,y_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2,y_2;\xi,\eta)d\xi d\eta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2-\xi,y_2-\eta)d\xi d\eta \\ &= g(x_2,y_2) ** h(x_2,y_2) \end{aligned}$$

where ** denotes 2D convolution. This will sometimes be abbreviated as *, e.g. $I(x_2, y_2) = g(x_2, y_2)*h(x_2, y_2)$.

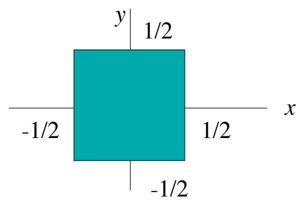
Rectangle Function

$$\Pi(x) = \begin{cases} 0 & |x| > 1/2 \\ 1 & |x| \le 1/2 \end{cases}$$

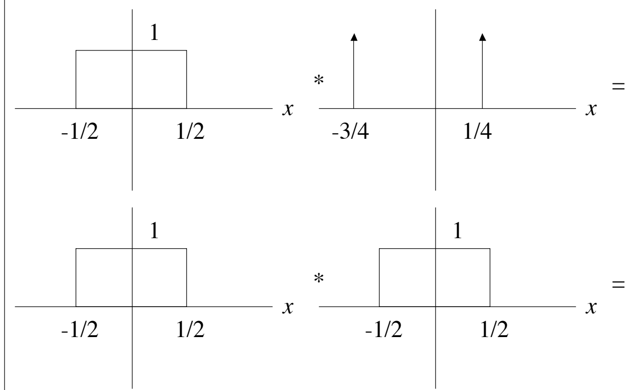


Also called rect(x)

$$\Pi(x,y) = \Pi(x)\Pi(y)$$

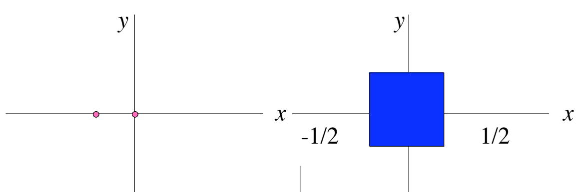


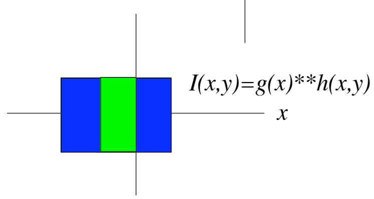
1D Convolution Examples



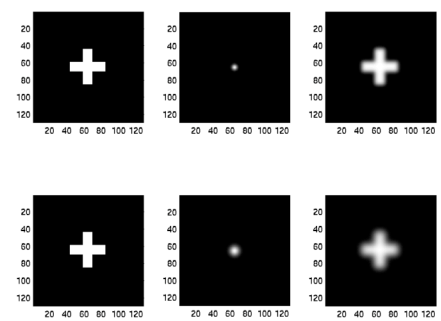
2D Convolution Example

g(x)= δ(x+1/2,y) + δ(x,y) *h(x)=rect(x,y)*





2D Convolution Example



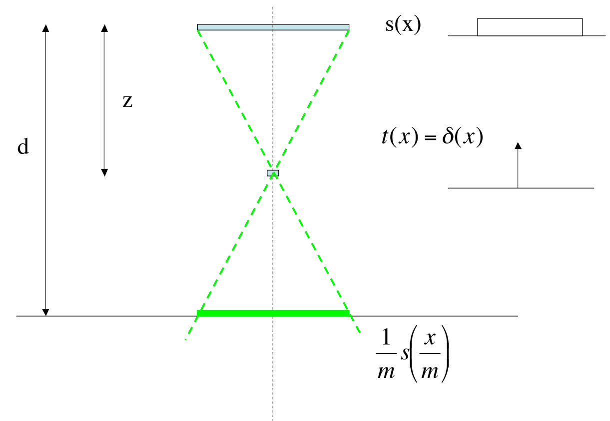
Pinhole Magnification Example

$$I(x_2,y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\xi,\eta)h(x_2,y_2;\xi,\eta)d\xi d\eta$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\xi,\eta)\delta(x_2 - m\xi, y_2 - m\eta)d\xi d\eta$$

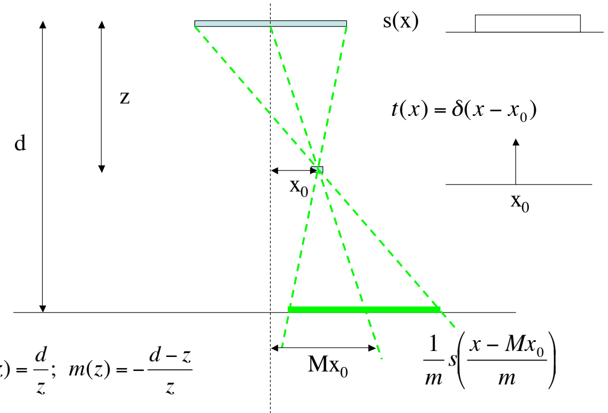
after substituting $\xi' = m\xi$ and $\eta' = m\eta$, we obtain

$$= \frac{1}{m^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\xi'/m,\eta'/m)\delta(x_2 - \xi', y_2 - \eta')d\xi' d\eta'$$
$$= \frac{1}{m^2} s(x_2/m, y_2/m) ** \delta(x_2,y_2)$$
$$= \frac{1}{m^2} s(x_2/m, y_2/m)$$

X-Ray Imaging



X-Ray Imaging



$M(z) = \frac{d}{z}; \quad m(z) = -\frac{d-z}{z}$

X-Ray Imaging

For off-center pinhole object, the shifted source image can be written as

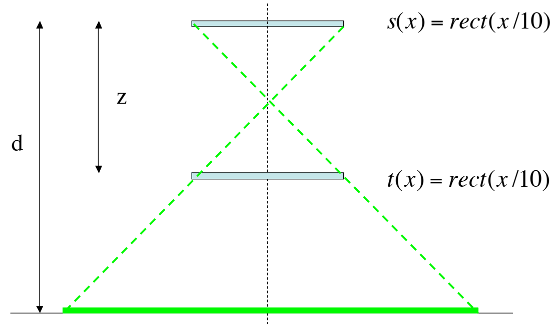
$$s\left(\frac{x - Mx_0}{m}\right) = s\left(\frac{x}{m}\right) * \delta\left(\frac{x}{M} - x_0\right)$$
$$= s(x/m) * t\left(\frac{x}{M}\right)$$

For the general 2D case, we convolve the magnified object with the impulse response

$$I(x,y) = t\left(\frac{x}{M},\frac{y}{M}\right) ** \frac{1}{m^2} s\left(\frac{x}{m},\frac{y}{m}\right)$$

Note: we have ignored obliquity factors etc.

X-Ray Imaging



$$\frac{1}{m}s\left(\frac{x}{m}\right) * t\left(\frac{x}{M}\right)$$

X-Ray Imaging

$m = 1; M = 2$

$$\frac{1}{m}s\left(\frac{x}{m}\right) * t\left(\frac{x}{M}\right) = rect(x/10) * rect(x/20)$$

$$= ???$$

Summary

1. The response to a linear system can be characterized by a spatially varying impulse response and the application of the superposition integral.
2. A shift invariant linear system can be characterized by its impulse response and the application of a convolution integral.