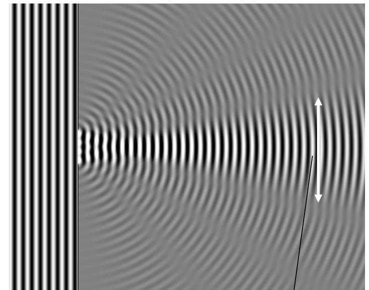
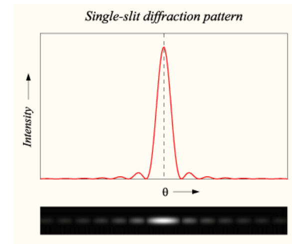
Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2009
Ultrasound Lecture 2

## Single-slit Diffraction

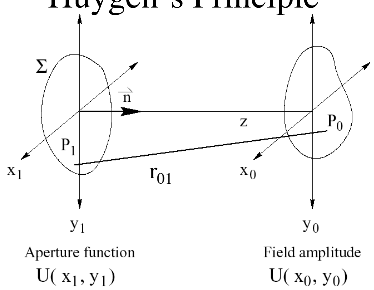





Beamwidth $\frac{\lambda z}{D}$

Source:wikipedia

## Huygen's Principle




*Wavenumber*

$k = \frac{2\pi}{\lambda}$

$$U(P_0) = \iint_{\Sigma} h(P_0, P_1) U(P_1) ds$$

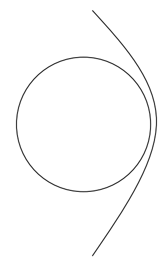
$$\text{where } h(P_0, P_1) = \frac{1}{j\lambda} \frac{\exp(jkr_{01})}{r_{01}} \cos(\vec{n}, \vec{r}_{01})$$

Oliquity Factor

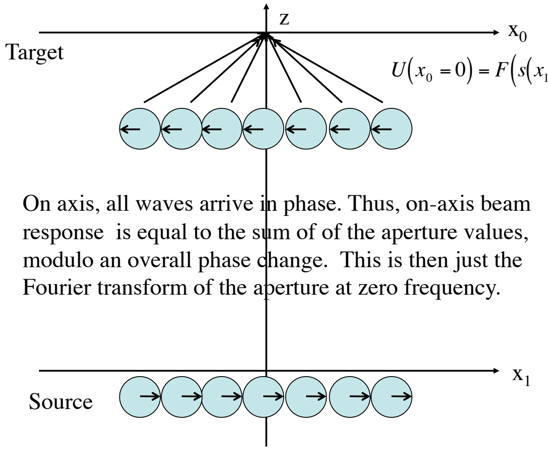
Anderson and Trahey 2000

## Plane-Wave Approximation

Approximates spherical wavefront
with a plane wave



# Plane Wave (Fraunhofer) Approximation

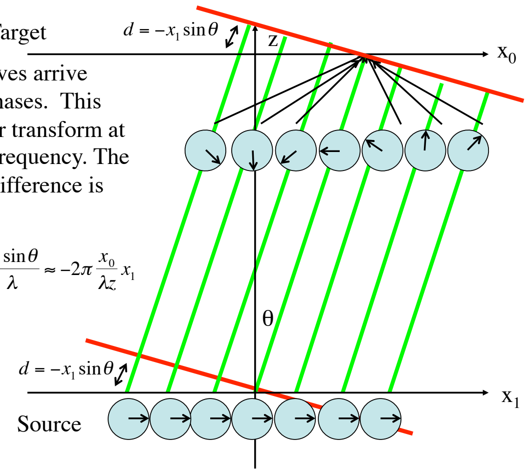



$$U\left(x_0 = 0\right) = F\left(s\left(x_1\right)\right)\Big|_{k_x=0}$$

On axis, all waves arrive in phase. Thus, on-axis beam response is equal to the sum of of the aperture values, modulo an overall phase change. This is then just the Fourier transform of the aperture at zero frequency.

Off axis, the waves arrive with different phases. This gives the Fourier transform at another spatial frequency. The relative phase difference is approximately

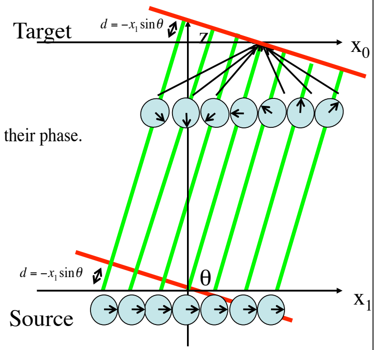
$$\Delta\varphi = 2\pi\frac{d}{\lambda} = -2\pi\frac{x_1\sin\theta}{\lambda} \approx -2\pi\frac{x_0}{\lambda z}x_1$$




$$\Delta\varphi = 2\pi\frac{d}{\lambda} = -2\pi\frac{x_1\sin\theta}{\lambda} \approx -2\pi\frac{x_0}{\lambda z}x_1$$

The signal at a point $x_0$ is given by integral over all source points $x_1$ weighted by their phase.

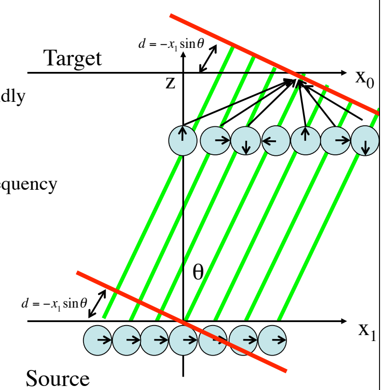
$$U\left(x_0\right) = \int_{-\infty}^{\infty} s\left(x_1\right)\exp\left(-j2\pi\frac{x_0}{\lambda z}x_1\right)dx_1$$
$$= \int_{-\infty}^{\infty} s\left(x_1\right)\exp\left(-j2\pi k_x x_1\right)dx_1$$
$$= F\left(s\left(x_1\right)\right)\Big|_{k_x=\frac{x_0}{\lambda z}}$$




As we increase $x_0$ (or equivalently $\theta$), the phase difference changes more rapidly as a function of the source position $x_1$.

This corresponds to a higher spatial frequency

$$k_x = \frac{x_0}{\lambda z}$$



# Plane Wave Approximation

In general

$$U(x_0,y_0)=\frac{1}{z}\exp\left(j\frac{2\pi z}{\lambda}\right)F\left[s(x,y)\right]_{k_x=\frac{x_0}{\lambda z},\,k_y=\frac{y_0}{\lambda z}}$$

*Example*

$$s(x,y)=rect(x/D)rect(y/D)$$

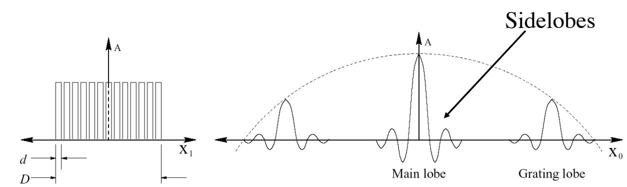
$$U(x_0,y_0)=\frac{1}{z}\exp(jkz)D^2\,\mathrm{sinc}\left(Dk_x\right)\mathrm{sinc}\left(Dk_y\right)$$

$$=\frac{1}{z}\exp(jkz)D^2\,\mathrm{sinc}\left(D\frac{x_0}{\lambda z}\right)\mathrm{sinc}\left(Dk_y\frac{y_0}{\lambda z}\right)$$

Zeros occur at $x_0=\frac{n\lambda z}{D}$ and $y_0=\frac{n\lambda z}{D}$

Beamwidth of the sinc function is $\frac{\lambda z}{D}$

# Example

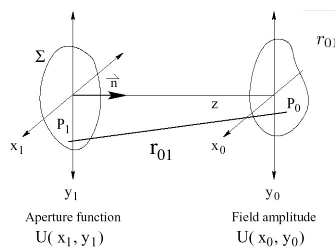



$$rect\left(\frac{x}{D}\right)\left[rect\left(\frac{x}{d}\right)*\frac{1}{d}comb\left(\frac{x}{d}\right)\right]\Leftrightarrow D\,\mathrm{sinc}(Dk_x)*\left[d\,\mathrm{sinc}(dk_x)comb(dk_x)\right]$$

Question: What should we do to reduce the sidelobes?

Anderson and Trahey 2000

# Small-Angle (paraxial) Approximation




$$r_{01}=\sqrt{z^2+(x_1-x_0)^2+(y_1-y_0)^2}$$

$$cos(\vec{n},\vec{r}_{01})\approx 1$$

$$r_{01}\approx z$$

$$h(x_0,y_0;x_1,y_1)\approx\frac{1}{j\lambda z}\exp(jkr_{01})$$

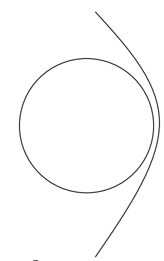
Anderson and Trahey 2000

# Fresnel Approximation

$$r_{01}=\sqrt{z^2+(x_1-x_0)^2+(y_1-y_0)^2}$$

$$=z\sqrt{1+\left(\frac{(x_1-x_0)}{z}\right)^2+\left(\frac{(y_1-y_0)^2}{z}\right)}$$

$$\approx z\left[1+\frac{1}{2}\left(\frac{(x_1-x_0)}{z}\right)^2+\frac{1}{2}\left(\frac{(y_1-y_0)}{z}\right)^2\right]$$

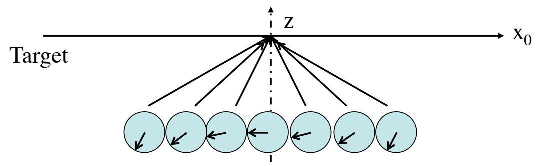


Approximates spherical wavefront with a parabolic phase profile

$$h(x_0,y_0;x_1,y_1)\approx\frac{\exp(jkz)}{j\lambda z}\exp\left[\frac{jk}{2z}\left[(x_1-x_0)^2+(y_1-y_0)^2\right]\right]$$

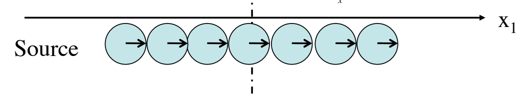
Anderson and Trahey 2000

# Fresnel Approximation




On axis, the signal is the integral of the source function $s(x_1)$ weighted by a quadratic phase term $\exp\left(\frac{\pi}{\lambda z}x_1^2\right)$

This is equivalent to $F\left(s(x_1)\exp\left(\frac{\pi}{\lambda z}x_1^2\right)\right)\Big|_{k_x=0}$




# Fresnel Zone

$$U(x_0,y_0) \approx \iint \frac{\exp(jkz)}{j\lambda z}\exp\left(\frac{jk}{2z}\left((x_1-x_0)^2+(y_1-y_0)^2\right)\right)s(x_1,y_1)dx_1dy_1$$

$$= \iint \frac{\exp(jkz)}{j\lambda z}\exp\left(\frac{jk}{2z}\left((x_1^2+y_1^2)+(x_0^2+y_0^2)-2(x_1x_0+y_1y_0)\right)\right)s(x_1,y_1)dx_1dy_1$$

$$= \frac{\exp(jkz)}{j\lambda z}\exp\left(\frac{jk}{2z}(x_0^2+y_0^2)\right)\times$$
$$\iint \frac{\exp(jkz)}{j\lambda z}\exp\left(\frac{jk}{2z}(x_1^2+y_1^2)\right)s(x_1,y_1)\exp\left(-\frac{jk}{z}(x_1x_0+y_1y_0)\right)dx_1dy_1$$

$$= \frac{\exp(jkz)}{j\lambda z}\exp\left(\frac{jk}{2z}(x_0^2+y_0^2)\right)F\left[\exp\left(\frac{jk}{2z}(x_1^2+y_1^2)\right)s(x_1,y_1)\right]$$

Phase across transducer face

# Fraunhofer Condition

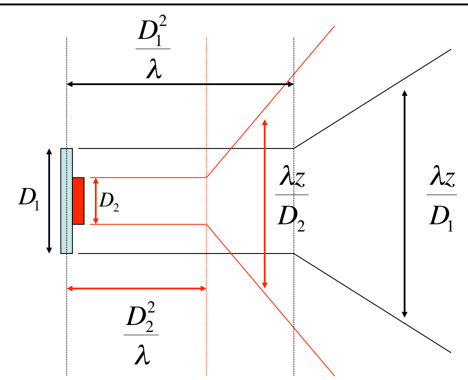
Phase term due to position on transducer is $\frac{k}{2z}(x_1^2+y_1^2)$

Far-field condition is

$$\frac{k}{2z}(x_1^2+y_1^2) << 1$$
$$z >> \frac{k}{2}(x_1^2+y_1^2) = \frac{\pi}{\lambda}(x_1^2+y_1^2)$$

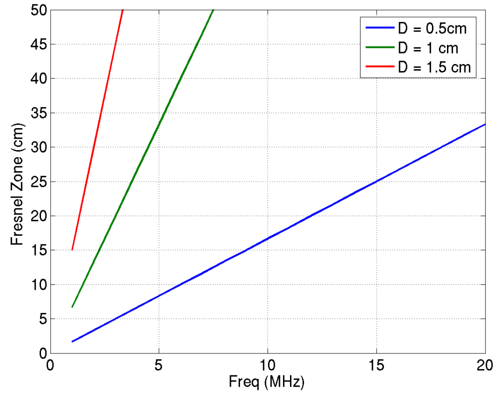
For a square DxD transducer, $x_1^2+y_1^2 = D^2/2$

$$z >> \frac{\pi D^2}{2\lambda} \approx \frac{D^2}{\lambda}$$



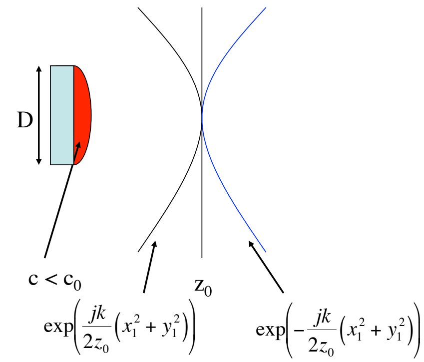

$$\left.\frac{\lambda z}{D}\right|_{z=D^2/\lambda} = D$$

## Fresnel Zone

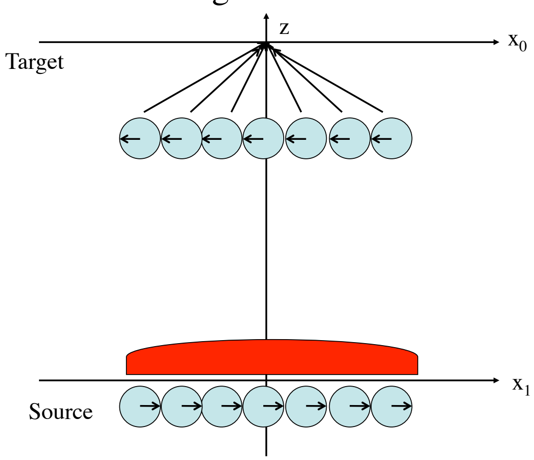



## Focusing in Fresnel Zone

$$U(x_0, y_0) = \frac{\exp(jkz)}{j\lambda z} \exp\left(\frac{jk}{2z}\left(x_0^2 + y_0^2\right)\right) F\left[\exp\left(\frac{jk}{2z}\left(x_1^2 + y_1^2\right)\right) s(x_1, y_1)\right]$$




## Focusing in Fresnel Zone




## Focusing in Fresnel Zone

$$U(x_0, y_0) = \frac{\exp(jkz)}{j\lambda z} \exp\left(\frac{jk}{2z}\left(x_0^2 + y_0^2\right)\right) F\left[\exp\left(\frac{jk}{2z}\left(x_1^2 + y_1^2\right)\right) s(x_1, y_1)\right]$$

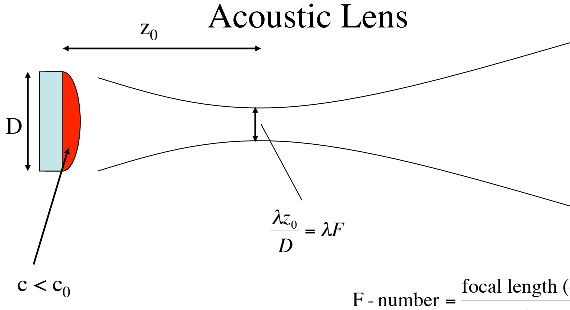
Make $s(x_1, y_1) = s_0(x_1, y_1) \exp\left(-\frac{jk}{2z_0}\left(x_1^2 + y_1^2\right)\right)$

At the focal depth $z = z_0$

$$U(x_0, y_0) = \frac{\exp(jkz_0)}{j\lambda z_0} \exp\left(\frac{jk}{2z_0}\left(x_0^2 + y_0^2\right)\right) F\left[s_0(x_1, y_1)\right]$$

Beamwidth at the focal depth is: $\frac{\lambda z_0}{D}$

# Acoustic Lens

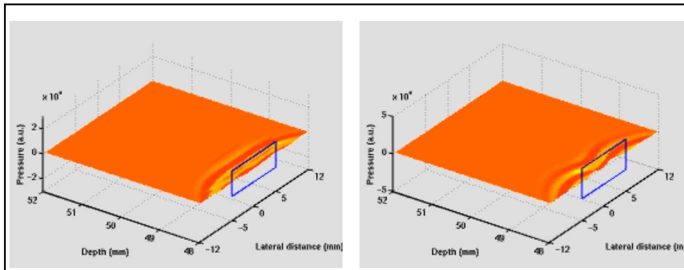


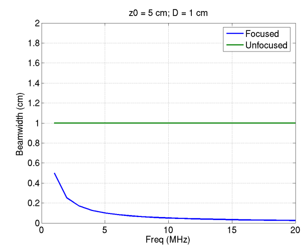

$$\text{F - number} = \frac{\text{focal length (depth)}}{\text{aperture size}}$$

At the focal depth $z = z_0$

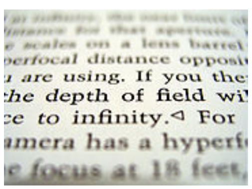
$$U(x_0, y_0) = \frac{\exp(jkz_0)}{j\lambda z_0} \exp\left(\frac{jk}{2z_0}\left(x_0^2 + y_0^2\right)\right) F\left[\exp\left(\frac{jk}{2z_0}\left(x_1^2 + y_1^2\right)\right)\exp\left(-\frac{jk}{2z_0}\left(x_1^2 + y_1^2\right)\right) s(x_1, y_1)\right]$$

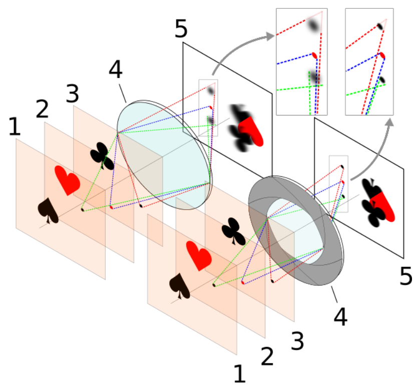
$$= \frac{\exp(jkz_0)}{j\lambda z_0} \exp\left(\frac{jk}{2z_0}\left(x_0^2 + y_0^2\right)\right) F\left[s(x_1, y_1)\right]$$




# Depth of Focus/Field






http://en.wikipedia.org/wiki/Depth_of_field

# Depth of Focus

When $z \neq z_0$, the phase term is $\Delta\Phi = \exp\left(-\frac{jk}{2z_0}\left(x_1^2 + y_1^2\right)\right)\exp\left(-\frac{jk}{2z}\left(x_1^2 + y_1^2\right)\right)$

and the lens is not perfectly focused.

Consider variation in the x - direction.

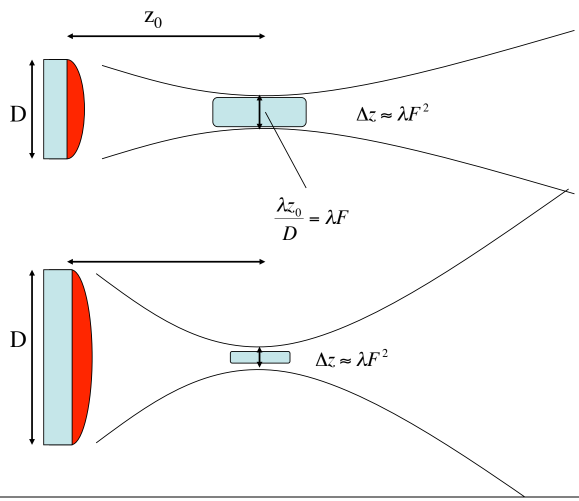
$$\Delta\Phi = \frac{kx^2}{2}\left(\frac{1}{z} - \frac{1}{z_0}\right)$$

For transducer of size D, $\frac{x^2}{2} = \frac{D^2}{8}$

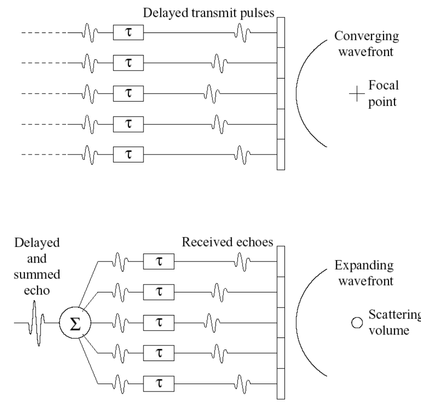
If we want $|\Delta\Phi| = \left|\frac{\pi D^2}{4\lambda}\left(\frac{1}{z} - \frac{1}{z_0}\right)\right| < 1$ radian then

$$\left|\frac{1}{z} - \frac{1}{z_0}\right| < \frac{4\lambda}{\pi D^2} \approx \frac{\lambda}{D^2} \Rightarrow \quad \frac{\Delta z}{z_0^2} < \frac{\lambda}{D^2}$$

$$\Delta z \approx \frac{\lambda z_0^2}{D^2} = \lambda F^2$$

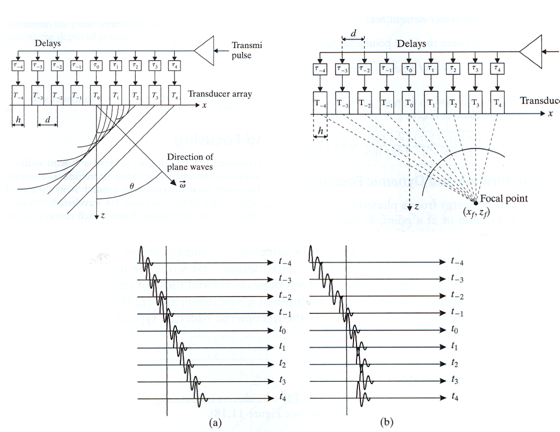



# Focusing with Phased Array



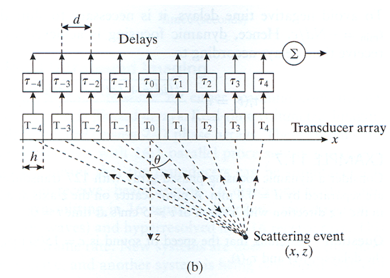

Anderson and Trahey 2000

# Focusing and Steering



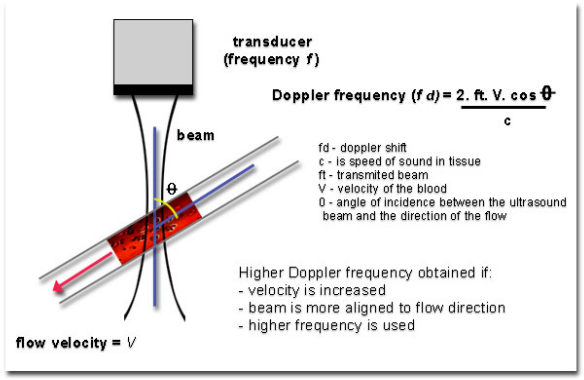

Prince and Links 2005

# Dynamic Focusing




Prince and Links 2005

## Doppler Effect




http://www.centrus.com.br/DiplomaFMF/SeriesFMF/doppler/capitulos-html/chapter_01.htm

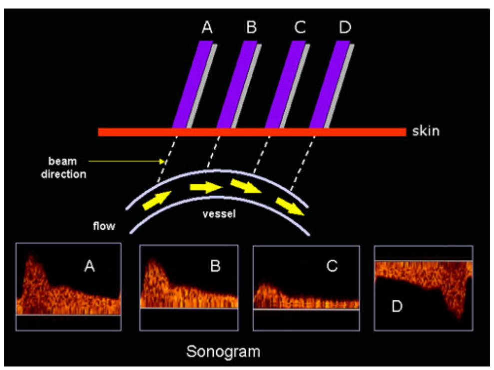
## Doppler Effect

$$\Delta f = \frac{2vf_0}{c-v} \approx \frac{2vf_0}{c}$$

Example
$v$ = 50 cm/s
c = 1500 m/s
f0 = 5 MHz

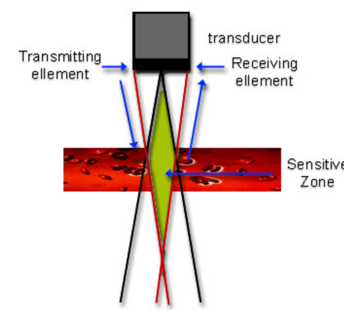
$$\frac{2vf_0}{c} = 3333\ Hz$$

## Doppler



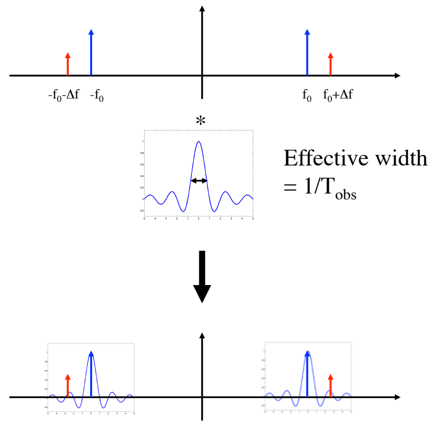

http://www.centrus.com.br/DiplomaFMF/SeriesFMF/doppler/capitulos-html/chapter_01.htm

## CW Doppler




http://www.centrus.com.br/DiplomaFMF/SeriesFMF/doppler/capitulos-html/chapter_01.htm

## CW Doppler




Effective width $= 1/T_{obs}$

## CW Doppler

Resolution $\Delta f = 1/T_{obs}$

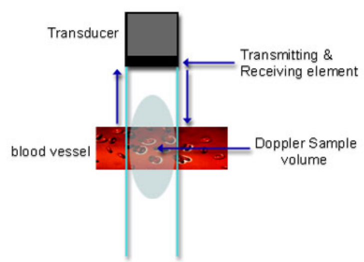
$$\Delta v = \frac{c\Delta f}{2f_0} = \frac{c}{2T_{obs}f_0}$$

Example
Design goal: $\Delta v = 5\ cm/s;\ f_0 = 5\ MHz$

$$T_{obs} = \frac{c}{2\Delta v f_0} = \frac{1500 m/s}{2(0.05 m/s)(5\times 10^6)} = 3ms$$

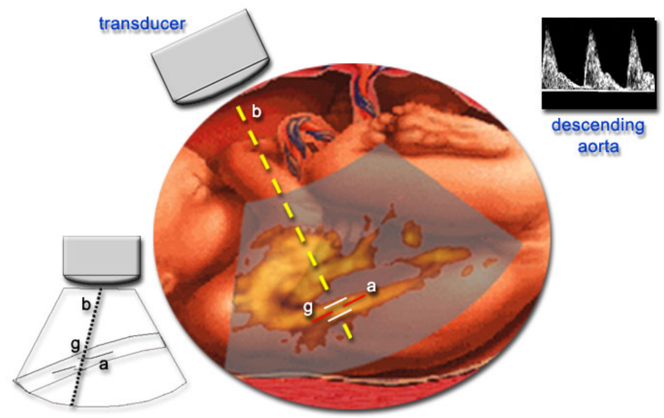
Note that for a depth of 15 cm, it takes only 200 usec for echos to return.

## PW Doppler

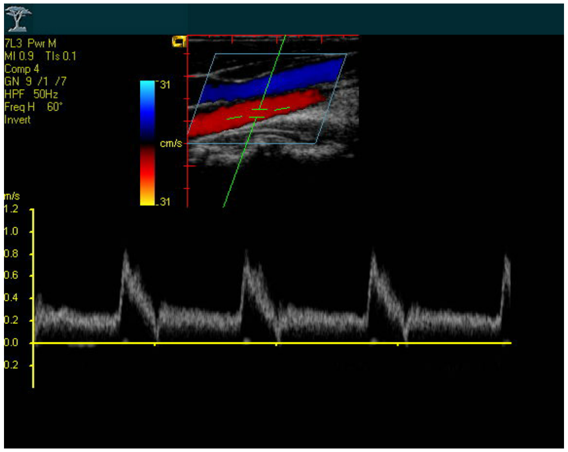



http://www.centrus.com.br/DiplomaFMF/SeriesFMF/doppler/capitulos-html/chapter_01.htm

## PW Doppler




http://www.centrus.com.br/DiplomaFMF/SeriesFMF/doppler/capitulos-html/chapter_01.htm




Siemens Medical Systems; jnormal common carotid artery

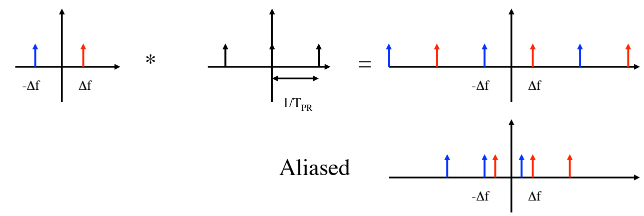
# Aliasing

Measure Doppler shifts at a specified range
For unambiguous range, one pulse at a time.

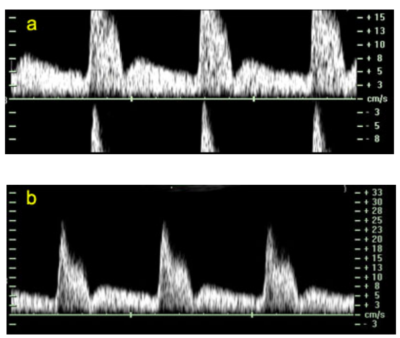
$T_{PR} = \frac{2r_{max}}{c}$ (e.g. 200 usec for 15 cm depth)

To avoid aliasing require

$\frac{1}{T_{PR}} > 2\Delta f_{max}$

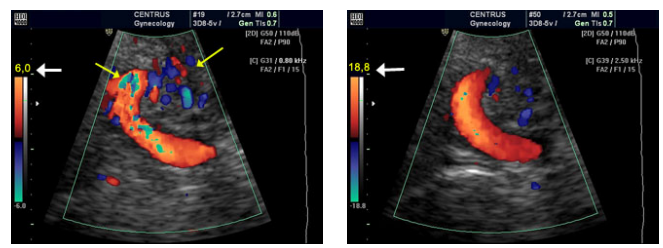



# Aliasing



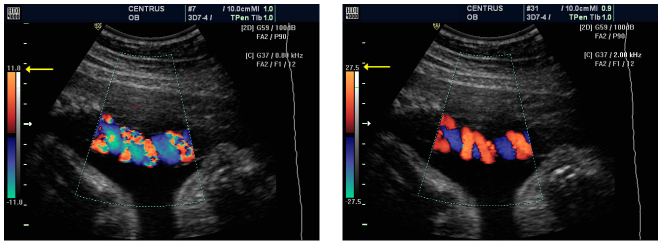

http://www.centrus.com.br/DiplomaFMF/SeriesFMF/doppler/capitulos-html/chapter_01.htm

# Aliasing




http://www.centrus.com.br/DiplomaFMF/SeriesFMF/doppler/capitulos-html/chapter_01.htm

## Aliasing



http://www.centrus.com.br/DiplomaFMF/SeriesFMF/doppler/capitulos-html/chapter_01.htm

## PW Doppler

Velocity Resolution (same as with CW)

$$\mathrm{T_{obs}} > \frac{1}{\Delta f} = \frac{c}{2\Delta v f_0}$$

Range Resolution

Want to interrogate velocities from a small region $\Delta z = \frac{cT_{pulse}}{2}$

We also need to make sure that particles remain within this region over the observation time $\mathrm{T_{obs}}$

$$v_{\max} T_{obs} < \Delta z \Rightarrow T_{obs} < \frac{\Delta z}{v_{\max}} = \frac{cT_{pulse}}{2v_{\max}}$$

## PW Doppler

Design Example

$$\mathrm{R_{max}} = 6 \text{ cm} \ \Rightarrow \ \mathrm{T_{PR}} = \frac{2(0.06m)}{1500m/s} = 80\ \mu\text{sec}$$

$$\frac{1}{\mathrm{T_{PR}}} > 2\Delta f_{\max} = \frac{4v_{\max} f_0}{c}$$

$$\frac{c}{4\mathrm{T_{PR}} f_0} > v_{\max} \Rightarrow \text{for } \mathrm{f_0} = 5MHz \ \text{ we find that } v_{\max} < 93.75cm/s$$

If we choose $\Delta v = 1cm/s$ then $\mathrm{T_{obs}} = \frac{c}{2\Delta v_{\max} f_0} = 15ms$

Range resolution : $\Delta z > v_{\max} T_{obs} = 1.4cm$

$$\mathrm{T_{pulse}} = \frac{2\Delta z}{c} = 18.8u\sec$$