Bioengineering 280A
Principles of Biomedical Imaging

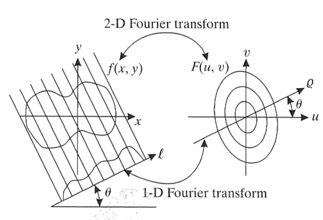
Fall Quarter 2009
CT/Fourier Lecture 2

Topics

- Projection Slice Theorem
- Fourier Transforms

Projection Slice Theorem

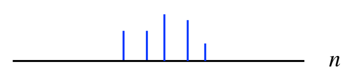
$$
\begin{aligned}
G(\rho,\theta) &= \int_{-\infty}^{\infty} g(l,\theta) e^{-j2\pi\rho l}\, dl \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - l) e^{-j2\pi\rho l}\, dx\, dy\, dl \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi\rho(x\cos\theta + y\sin\theta)}\, dx\, dy \\
&= F_{2D}\left[f(x,y)\right]\Big|_{u=\rho\cos\theta, v=\rho\sin\theta}
\end{aligned}
$$

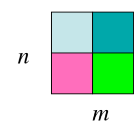


Prince&Links 2006

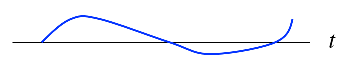
Signals and Images

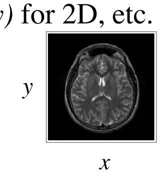
Discrete-time/space signal/image: continuous valued function with a discrete time/space index, denoted as *s[n]* for 1D, *s[m,n]* for 2D , etc.





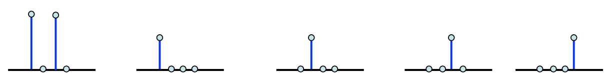
Continuous-time/space signal/image: continuous valued function with a continuous time/space index, denoted as *s(t)* or *s(x)* for 1D, *s(x,y)* for 2D, etc.



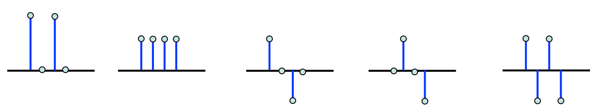


1D Signal Decomposition

$$\{2,0,2,0\} = 2\cdot\{1,0,0,0\} + 0\cdot\{0,1,0,0\} + 2\cdot\{0,0,1,0\} + 0\cdot\{0,0,0,1\}$$



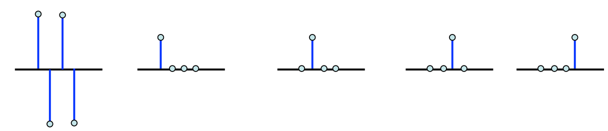
$$\{2,0,2,0\} = a\cdot\{1,1,1,1\} + b\cdot\{1,0,-1,0\} + c\cdot\{0,1,0,-1\} + d\cdot\{1,-1,1,-1\}$$



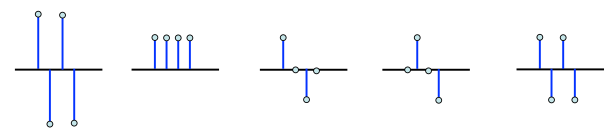
$$\{2,0,2,0\} = 1\cdot\{1,1,1,1\} + 0\cdot\{1,0,-1,0\} + 0\cdot\{0,1,0,-1\} + 1\cdot\{1,-1,1,-1\}$$

1D Signal Decomposition

$$\{2,-2,2,-2\} = 2\cdot\{1,0,0,0\} - 2\cdot\{0,1,0,0\} + 2\cdot\{0,0,1,0\} - 2\cdot\{0,0,0,1\}$$



$$\{2,-2,2,-2\} = 0\cdot\{1,1,1,1\} + 0\cdot\{1,0,-1,0\} + 0\cdot\{0,1,0,-1\} + 2\cdot\{1,-1,1,-1\}$$



Eskimo Words for Snow

tlapa	powder snow
tlacringit	snow that is crusted on the surface
kayi	drifting snow
tlapat	still snow
klin	remembered snow
naklin	forgotten snow
tlamo	snow that falls in large wet flakes
tlatim	snow that falls in small flakes
tlaslo	snow that falls slowly
tlapinti	snow that falls quickly
kripya	snow that has melted and refrozen
tliyel	snow that has been marked by wolves
tliyelin	snow that has been marked by Eskimos
tlalman	snow sold to German tourists
tlalam	snow sold to American tourists
tlanip	snow sold to Japanese tourists
tla-na-na	snow mixed with the sound of old rock and roll from a portable radio
depptla	a small snowball, preserved in Lucite, that had been handled by Johnny Depp

http://www.mendosa.com/snow.html

Image Compression

2D Image

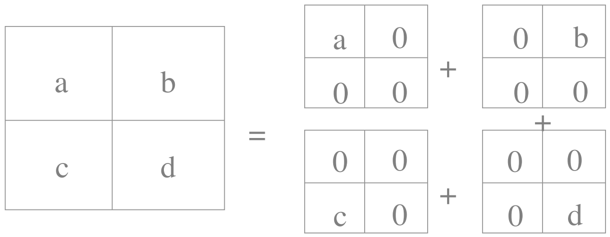
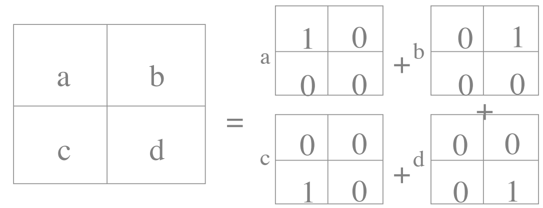
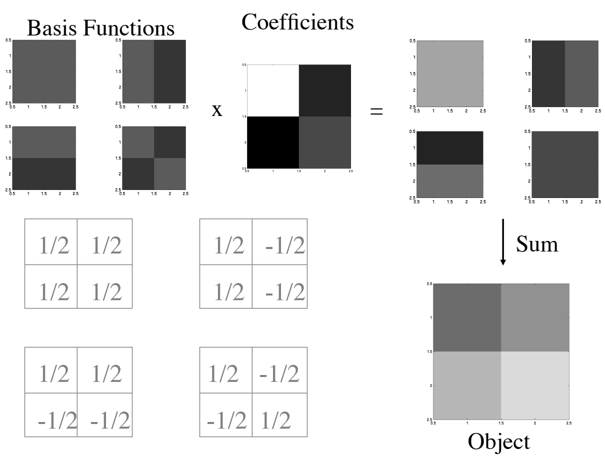
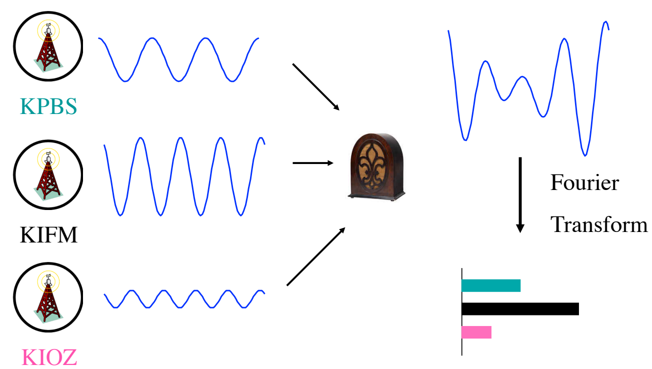


Image Decomposition





1D Fourier Transform



The Fourier Transform

Fourier Transform (FT)

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt = F\{g(t)\}$$

Inverse Fourier Transform

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df = F^{-1}\{G(f)\}$$

Complex Numbers

$j = \sqrt{-1}$

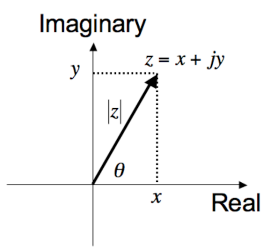
$j^2 = ?$

$(3+2j)(3-2j) = ?$

$j^2 = -1$

$(3+2j)(3-2j) = 9 - 4j^2 = 13$

Complex Numbers



$z = 2 + 1j$

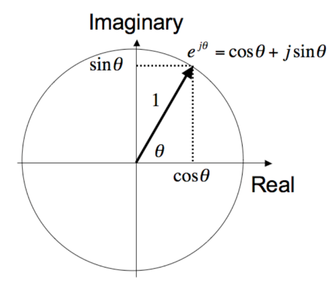
$|z| = \sqrt{2^2 + 1} = \sqrt{5}$

$\theta = \tan^{-1}\left(\frac{1}{2}\right) = 30$ degrees

Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$z = x + jy = |z| e^{j\theta}$$



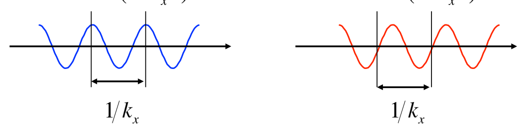
1D Fourier Transform

$$G(k_x) = \int_{-\infty}^{\infty} g(x)\exp(-j2\pi k_x x)dx$$

$$= \int_{-\infty}^{\infty} g(x)\cos(2\pi k_x x)dx - j\int_{-\infty}^{\infty} g(x)\sin(2\pi k_x x)dx$$

The part of $g(x)$ that "looks" like $\cos(2\pi k_x x)$

The part of $g(x)$ that "looks" like $\sin(2\pi k_x x)$



Units

Temporal Coordinates, e.g. t in seconds, f in cycles/second

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt \qquad \text{Fourier Transform}$$

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df \qquad \text{Inverse Fourier Transform}$$

Spatial Coordinates, e.g. x in cm, k_x is spatial frequency in cycles/cm

$$G(k_x) = \int_{-\infty}^{\infty} g(x)e^{-j2\pi k_x x}dx \qquad \text{Fourier Transform}$$

$$g(x) = \int_{-\infty}^{\infty} G(k_x)e^{j2\pi k_x x}dk_x \qquad \text{Inverse Fourier Transform}$$

2D Fourier Transform

Fourier Transform

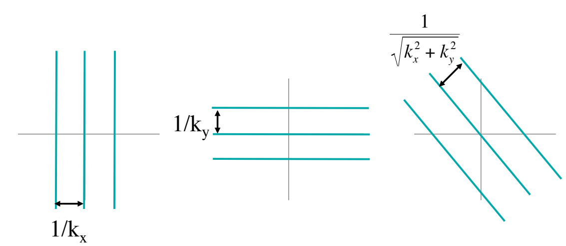
$$G(k_x,k_y) = F\left[g(x,y)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi(k_x x + k_y y)}dxdy$$

Inverse Fourier Transform

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x,k_y)e^{j2\pi(k_x x + k_y y)}dk_x dk_y$$

Plane Waves

$$e^{j2\pi(k_x x + k_y y)} = \cos\left(2\pi(k_x x + k_y y)\right) + j\sin\left(2\pi(k_x x + k_y y)\right)$$



$\cos(2\pi k_x x)$ $\qquad$ $\cos(2\pi k_y y)$ $\qquad$ $\cos(2\pi k_x x + 2\pi k_y y)$

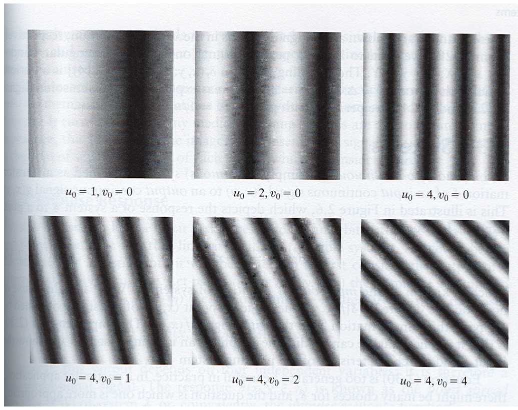
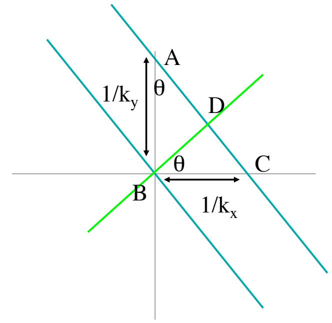


Figure 2.5 from Prince and Link

Plane Waves



$\Delta ABC \sim \Delta BDC$

$$\frac{AC}{BC} = \frac{AB}{BD}$$

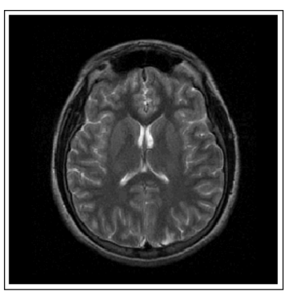
$$BD = AB\frac{BC}{AC} = \frac{\frac{1}{k_x}\frac{1}{k_y}}{\sqrt{\frac{1}{k_x^2}+\frac{1}{k_y^2}}} = \frac{1}{\sqrt{k_x^2+k_y^2}}$$

$$\theta = \arctan\left(\frac{k_y}{k_x}\right)$$

k-space

Image space

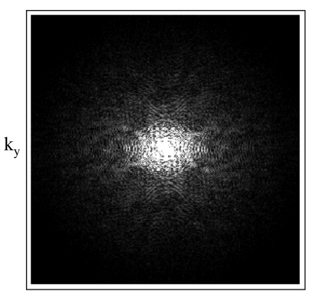
y



x

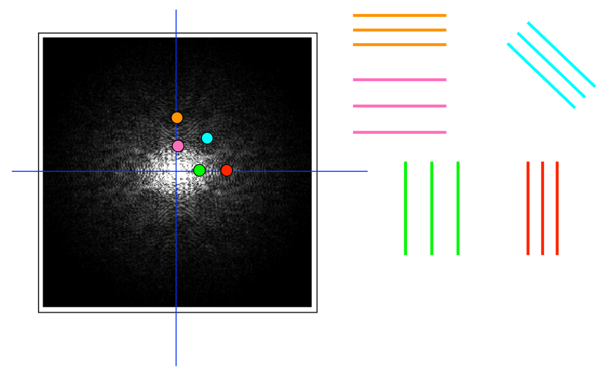
k-space

k_y

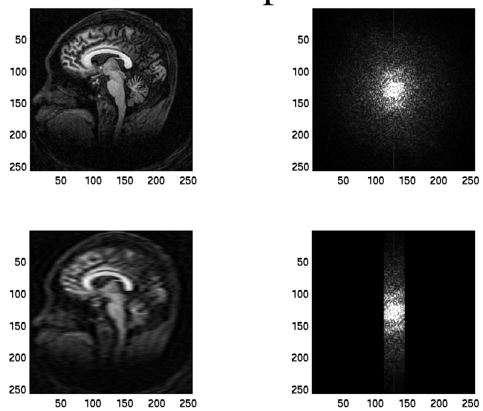


k_x

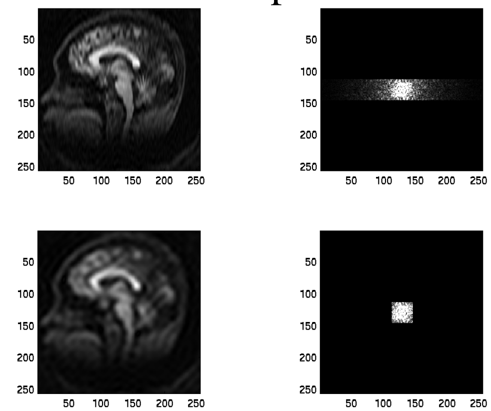
Fourier Transform



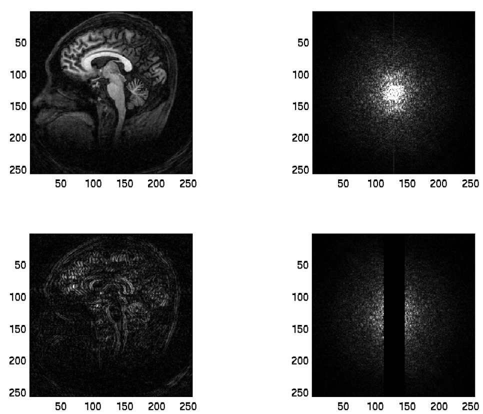
Examples



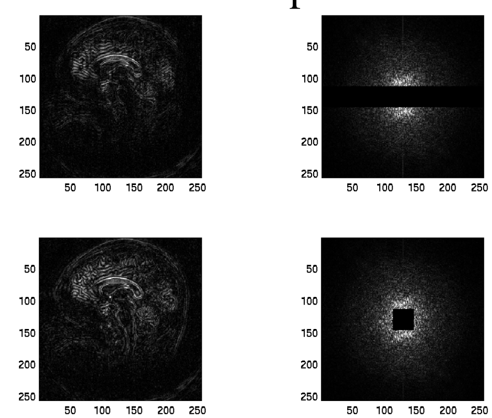
Examples



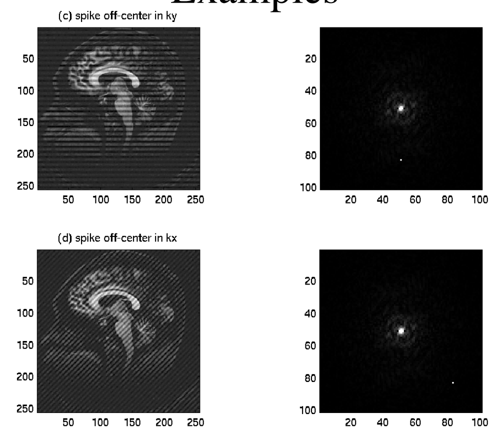
Examples

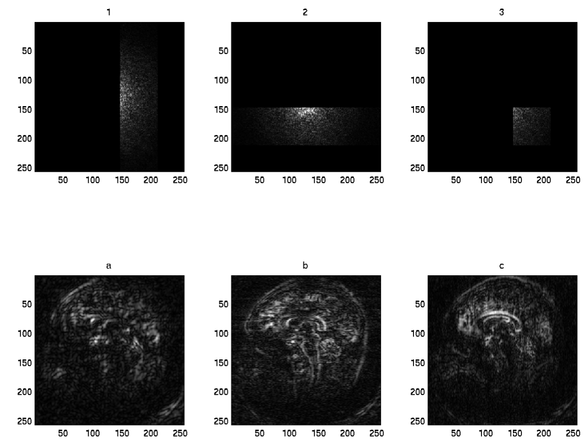


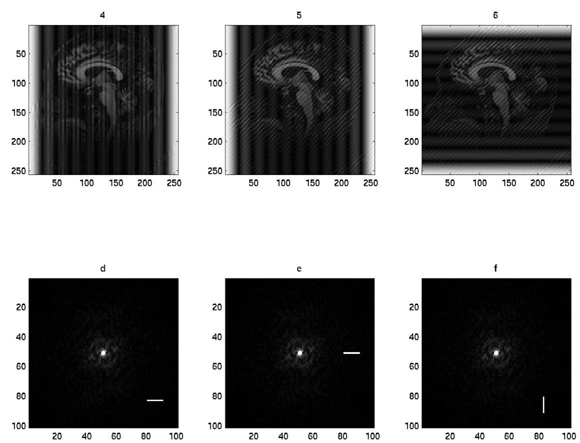
Examples



Examples


(c) spike off-center in ky
(d) spike off-center in kx


1
2
3
a
b
c


4
5
6
d
e
f