

Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2009
CT/Fourier Lecture 3

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Computing Transforms

$$F(\delta(x)) = \int_{-\infty}^{\infty} \delta(x) e^{-j2\pi k_x x} dx = 1$$

$$F(\delta(x - x_0)) = \int_{-\infty}^{\infty} \delta(x - x_0) e^{-j2\pi k_x x} dx = e^{-j2\pi k_x x_0}$$

$$\begin{aligned} F(\Pi(x)) &= \int_{-1/2}^{1/2} e^{-j2\pi k_x x} dx \\ &= \frac{e^{-j\pi k_x} - e^{j\pi k_x}}{-j2\pi k_x} \\ &= \frac{\sin(\pi k_x)}{\pi k_x} = \text{sinc}(k_x) \end{aligned}$$

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Computing Transforms

$$F(1) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx = ???$$

Define $h(k_x) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx$ and see what it does under an integral.

$$\begin{aligned} \int_{-\infty}^{\infty} G(k_x) h(k_x) dk_x &= \int_{-\infty}^{\infty} G(k_x) \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx dk_x \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(k_x) e^{-j2\pi k_x x} dk_x dx \\ &= \int_{-\infty}^{\infty} g(-x) dx \\ &= G(0) \end{aligned}$$

$$\text{Therefore, } F(1) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx = \delta(k_x)$$

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Computing Transforms

Similarly,

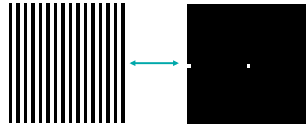
$$\begin{aligned} F\{e^{j2\pi k_0 x}\} &= \delta(k_x - k_0) \\ F\{\cos 2\pi k_0 x\} &= \frac{1}{2}(\delta(k_x - k_0) + \delta(k_x + k_0)) \\ F\{\sin 2\pi k_0 x\} &= \frac{1}{2j}(\delta(k_x - k_0) - \delta(k_x + k_0)) \end{aligned}$$

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Examples

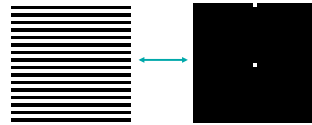
$$g(x, y) = 1 + e^{-j2\pi ax}$$

$$G(k_x, k_y) = \delta(k_x, k_y) + \delta(k_x + a)\delta(k_y)$$



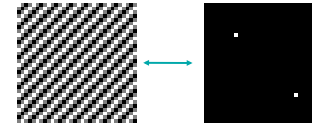
$$g(x, y) = 1 + e^{j2\pi by}$$

$$G(k_x, k_y) = \delta(k_x, k_y) + \delta(k_y)\delta(k_y - a)$$



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Examples

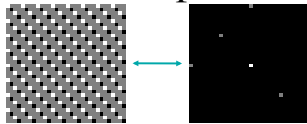


$$g(x, y) = \cos(2\pi(ax + by))$$

$$G(k_x, k_y) = \frac{1}{2}\delta(k_x - a)\delta(k_y - b) + \frac{1}{2}\delta(k_x + a)\delta(k_y + b)$$

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Examples



$$G(k_x, k_y) = \delta(k_x, k_y) + \delta(k_x + c)\delta(k_y) + \delta(k_y)\delta(k_y - d) + \frac{1}{2}\delta(k_x - a)\delta(k_y - b) + \frac{1}{2}\delta(k_x + a)\delta(k_y + b)$$

$$g(x, y) = ???$$

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Basic Properties

Linearity

$$F[ag(x, y) + bh(x, y)] = aG(k_x, k_y) + bH(k_x, k_y)$$

Scaling

$$F[g(ax, by)] = \frac{1}{|ab|} G\left(\frac{k_x}{a}, \frac{k_y}{b}\right)$$

Shift

$$F[g(x - a, y - b)] = G(k_x, k_y)e^{-j2\pi(k_x a + k_y b)}$$

Modulation

$$F[g(x, y)e^{j2\pi(xa + yb)}] = G(k_x - a, k_y - b)$$

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Linearity

The Fourier Transform is linear.

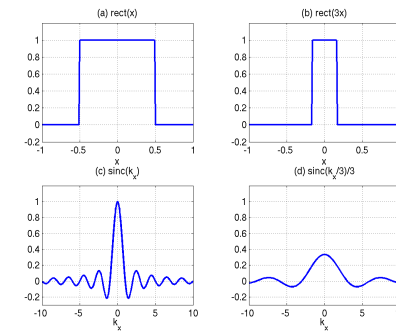
$$F\{ag(x) + bh(x)\} = aG(k_x) + bH(k_x)$$

$$F[ag(x,y) + bh(x,y)] = aG(k_x, k_y) + bH(k_x, k_y)$$

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Scaling Theorem

$$F\{g(ax)\} = \frac{1}{|a|}G\left(\frac{k_x}{a}\right) \quad F[g(ax, by)] = \frac{1}{|ab|}G\left(\frac{k_x}{a}, \frac{k_y}{b}\right)$$



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Separable Functions

$g(x, y)$ is said to be a separable function if it can be written as $g(x, y) = g_x(x)g_y(y)$

The Fourier Transform is then separable as well.

$$\begin{aligned} G(k_x, k_y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{-j2\pi(k_x x + k_y y)} dx dy \\ &= \int_{-\infty}^{\infty} g_x(x) e^{-j2\pi k_x x} dx \int_{-\infty}^{\infty} g_y(y) e^{-j2\pi k_y y} dy \\ &= G_x(k_x) G_y(k_y) \end{aligned}$$

Example

$$g(x, y) = \Pi(x)\Pi(y)$$

$$G(k_x, k_y) = \text{sinc}(k_x)\text{sinc}(k_y)$$

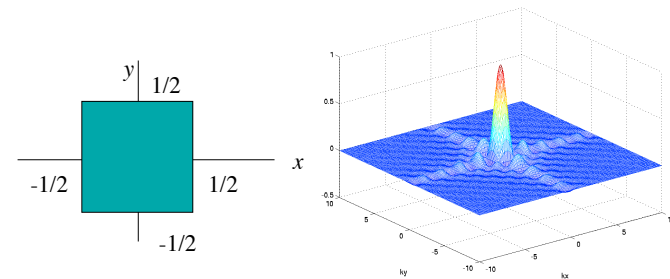
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Example (sinc/rect)

Example

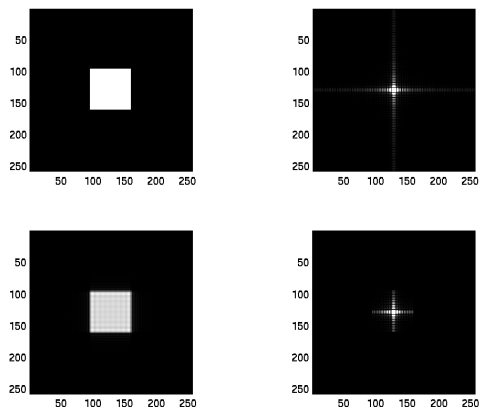
$$g(x, y) = \Pi(x)\Pi(y)$$

$$G(k_x, k_y) = \text{sinc}(k_x)\text{sinc}(k_y)$$



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Example (sinc/rect)



Examples

$$g(x, y) = \delta(x, y) = \delta(x)\delta(y)$$

$$G(k_x, k_y) = 1$$

$$g(x, y) = \delta(x)$$

$$G(k_x, k_y) = \delta(k_y) \quad !!!$$

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Duality

Note the similarity between these two transforms

$$F\{e^{j2\pi ax}\} = \delta(k_x - a)$$

$$F\{\delta(x - a)\} = e^{-j2\pi ka}$$

These are specific cases of duality

$$F\{G(x)\} = g(-k_x)$$

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Application of Duality

$$F\{\text{sinc}(x)\} = \int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x} e^{-j2\pi k_x x} dx = ??$$

Recall that $F\{\Pi(x)\} = \text{sinc}(k_x)$.

Therefore from duality, $F\{\text{sinc}(x)\} = \Pi(-k_x) = \Pi(k_x)$

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Shift Theorem

$$F\{g(x-a)\} = G(k_x)e^{-j2\pi ak_x}$$

$$F[g(x-a, y-b)] = G(k_x, k_y)e^{-j2\pi(k_x a + k_y b)}$$

Shifting the function doesn't change its spectral content, so the magnitude of the transform is unchanged.

Each frequency component is shifted by a . This corresponds to a relative phase shift of

$$-2\pi a / (\text{spatial period}) = -2\pi ak_x$$

For example, consider $\exp(j2\pi k_x x)$. Shifting this by a yields $\exp(j2\pi k_x (x-a)) = \exp(j2\pi k_x x) \exp(-j2\pi ak_x)$

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Modulation

$$F[g(x)e^{j2\pi k_0 x}] = G(k_x) * \delta(k_x - k_0) = G(k_x - k_0)$$

$$F[g(x)\cos(2\pi k_0 x)] = \frac{1}{2}G(k_x - k_0) + \frac{1}{2}G(k_x + k_0)$$

$$F[g(x)\sin(2\pi k_0 x)] = \frac{1}{2j}G(k_x - k_0) - \frac{1}{2j}G(k_x + k_0)$$

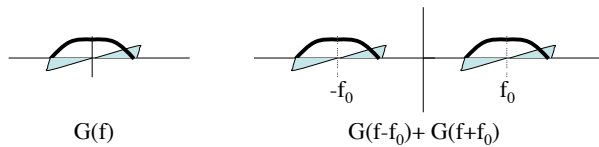
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Example

Amplitude Modulation (e.g. AM Radio)

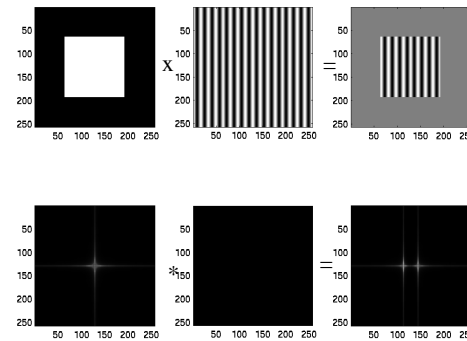
$$g(t) \rightarrow 2g(t)\cos(2\pi f_0 t)$$

$$2\cos(2\pi f_0 t)$$

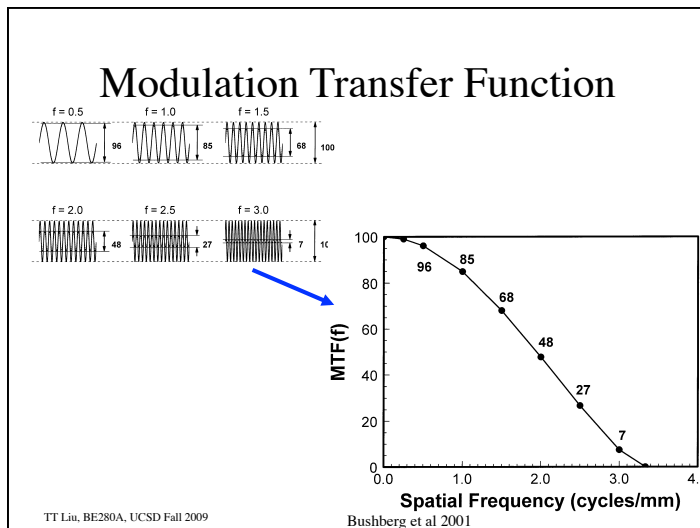
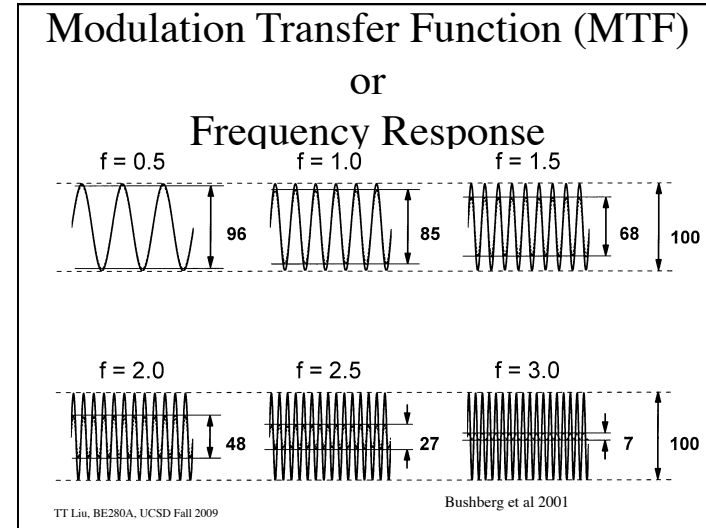
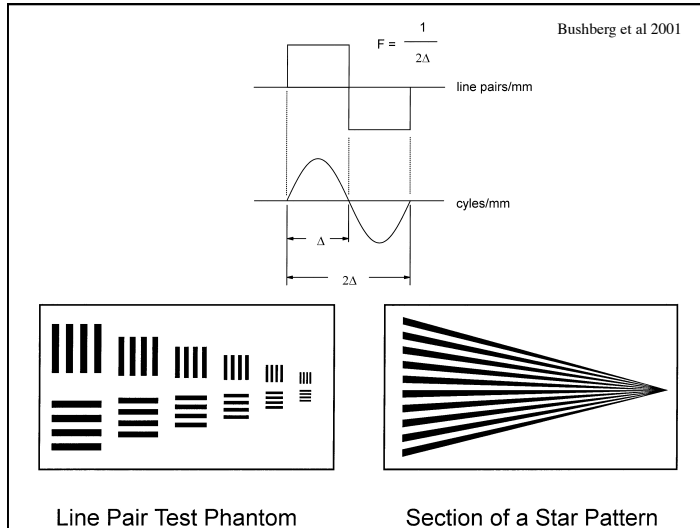


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Modulation Example



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Eigenfunctions

The fundamental nature of the convolution theorem may be better understood by observing that the complex exponentials are eigenfunctions of the convolution operator.

$$e^{j2\pi k_x x} \longrightarrow \boxed{g(x)} \longrightarrow z(x)$$

$$z(x) = g(x) * e^{j2\pi k_x x}$$

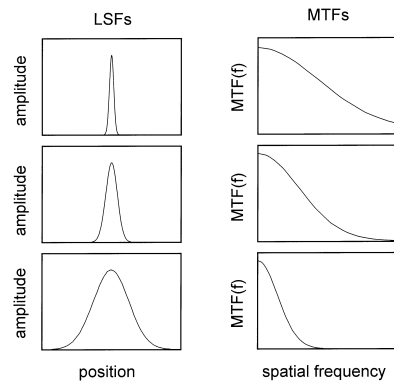
$$= \int_{-\infty}^{\infty} g(u) e^{j2\pi k_x (x-u)} du$$

$$= G(k_x) e^{j2\pi k_x x}$$

The response of a linear shift invariant system to a complex exponential is simply the exponential multiplied by the FT of the system's impulse response.

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MTF = Fourier Transform of PSF

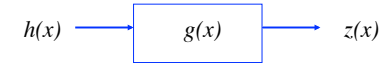


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Bushberg et al 2001

Convolution/Multiplication

Now consider an arbitrary input $h(x)$.



Recall that we can express $h(x)$ as the integral of weighted complex exponentials.

$$h(x) = \int_{-\infty}^{\infty} H(k_x) e^{j2\pi k_x x} dk_x$$

Each of these exponentials is weighted by $G(k_x)$ so that the response may be written as

$$z(x) = \int_{-\infty}^{\infty} G(k_x) H(k_x) e^{j2\pi k_x x} dk_x$$

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Convolution/Modulation Theorem

$$\begin{aligned} F\{g(x) * h(x)\} &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g(u) * h(x-u) du \right] e^{-j2\pi k_x x} dx \\ &= \int_{-\infty}^{\infty} g(u) \int_{-\infty}^{\infty} h(x-u) e^{-j2\pi k_x x} dx du \\ &= \int_{-\infty}^{\infty} g(u) H(k_x) e^{-j2\pi k_x u} du \\ &= G(k_x) H(k_x) \end{aligned}$$

Convolution in the spatial domain transforms into multiplication in the frequency domain. Dual is modulation

$$F\{g(x)h(x)\} = G(k_x) * H(k_x)$$

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2D Convolution/Multiplication

Convolution

$$F[g(x, y) ** h(x, y)] = G(k_x, k_y) H(k_x, k_y)$$

Multiplication

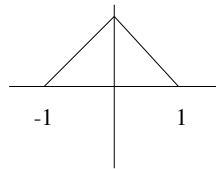
$$F[g(x, y)h(x, y)] = G(k_x, k_y) ** H(k_x, k_y)$$

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Application of Convolution Thm.

$$\Lambda(x) = \begin{cases} 1-|x| & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F(\Lambda(x)) = \int_{-1}^1 (1-|x|) e^{-j2\pi k_x x} dx = ??$$

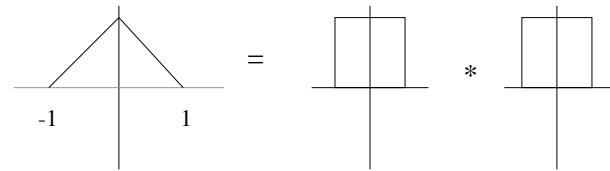


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Application of Convolution Thm.

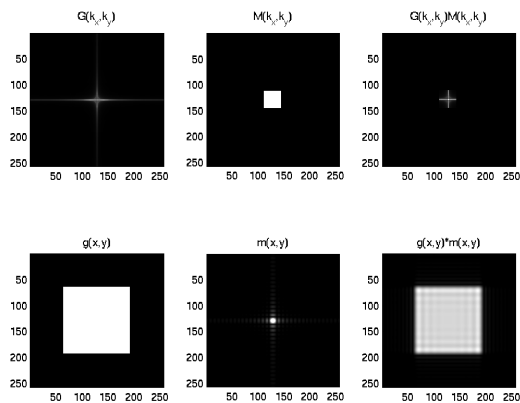
$$\Lambda(x) = \Pi(x) * \Pi(x)$$

$$F(\Lambda(x)) = \text{sinc}^2(k_x)$$

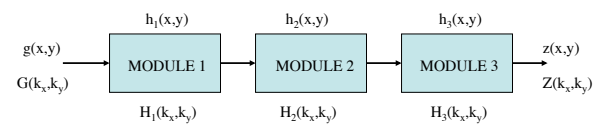


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Convolution Example



Response of an Imaging System

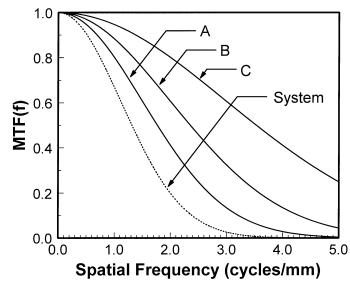


$$z(x, y) = g(x, y) ** h_1(x, y) ** h_2(x, y) ** h_3(x, y)$$

$$Z(k_x, k_y) = G(k_x, k_y) H_1(k_x, k_y) H_2(k_x, k_y) H_3(k_x, k_y)$$

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System MTF = Product of MTFs of Components



Bushberg et al 2001

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Useful Approximation

$$FWHM_{System} = \sqrt{FWHM_1^2 + FWHM_2^2 + \dots + FWHM_N^2}$$

Example

$$FWHM_1 = 1 \text{ mm}$$

$$FWHM_2 = 2 \text{ mm}$$

$$FWHM_{System} = \sqrt{5} = 2.24 \text{ mm}$$

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Figure 1:

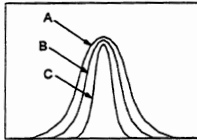
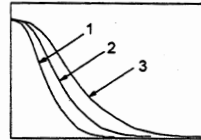


Figure 2:



8. Referring to Figure 1 (above) which demonstrates 3 different line spread functions (LSF), which LSF will yield the best spatial resolution?
 10. Referring to Figure 1 which shows LSFs, and Figure 2 which shows the corresponding modulation transfer functions (MTFs), which MTF corresponds to LSF C?
 - A. MTF number 1
 - B. MTF number 2
 - C. MTF number 3
- D74.** The intrinsic resolution of a gamma camera is 5 mm. The collimator resolution is 10 mm. The overall system resolution is ____ mm.
- A. 15
 - B. 11.2
 - C. 7.5
 - D. 5.0
 - E. 0.5