

Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2009
CT/Fourier Lecture 5

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Topics

- Sampling Requirements in CT
- Sampling Theory
- Aliasing

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CT Sampling Requirements

What should the size of the detectors be?

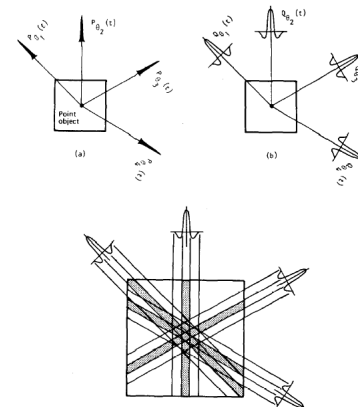
How many detectors do we need?

How many views do we need?

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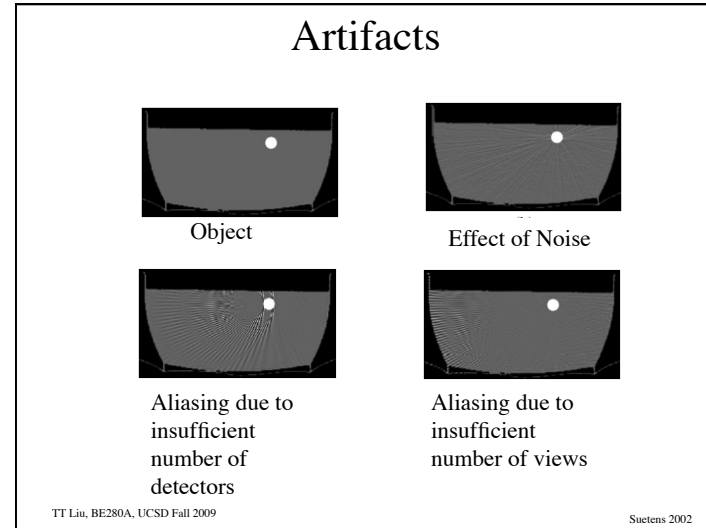
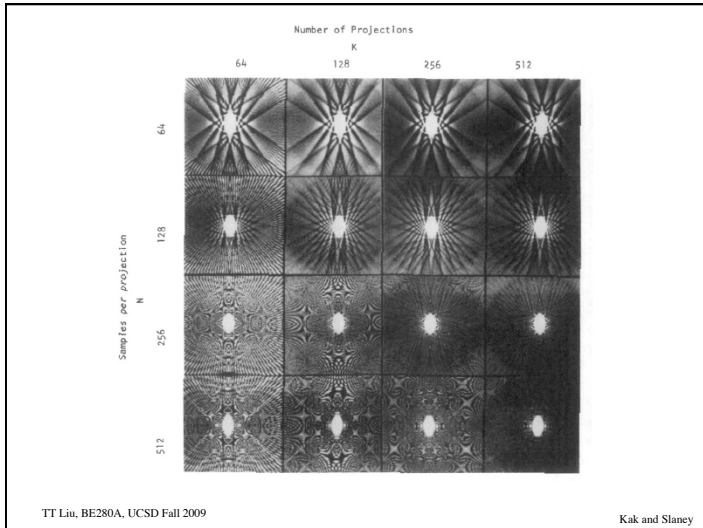
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View Aliasing



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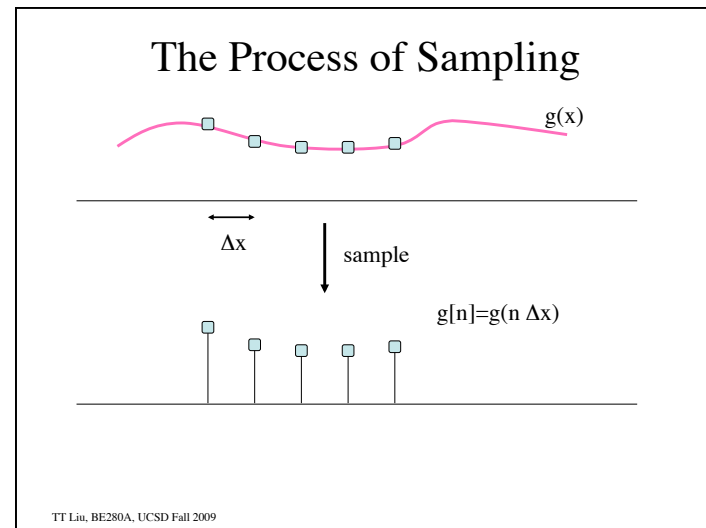


Analog vs. Digital

The Analog World:
Continuous time/space, continuous valued signals or images, e.g. vinyl records, photographs, x-ray films.

The Digital World:
Discrete time/space, discrete-valued signals or images, e.g. CD-Roms, DVDs, digital photos, digital x-rays, CT, MRI, ultrasound.

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Questions

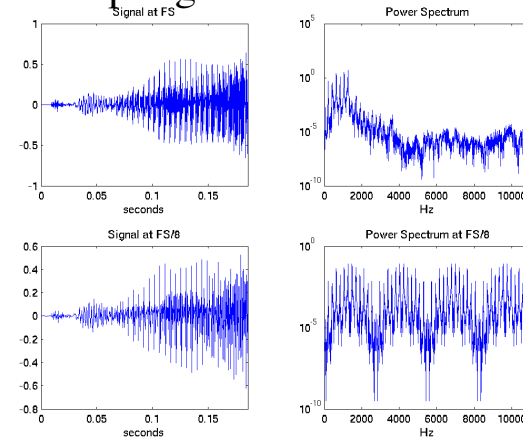
How finely do we need to sample?

What happens if we don't sample finely enough?

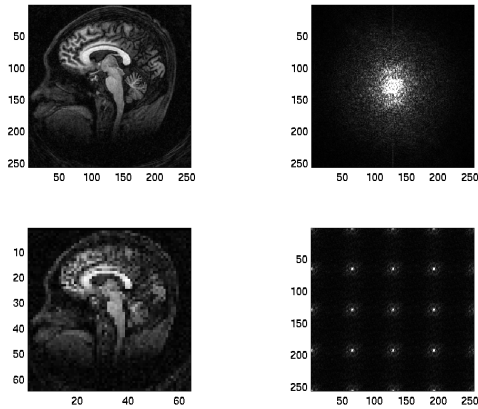
Can we reconstruct the original signal or image from its samples?

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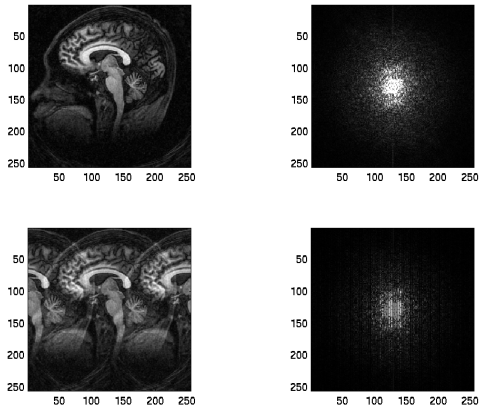
Sampling in the Time Domain



Sampling in Image Space

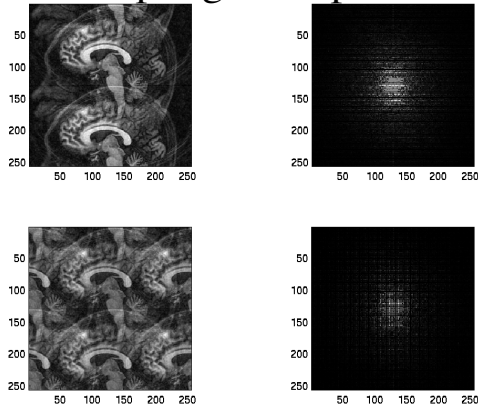


Sampling in k-space



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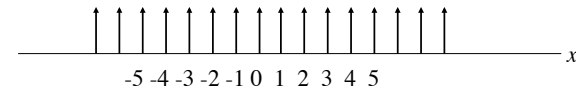
Sampling in k-space



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Comb Function

$$\text{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x - n)$$

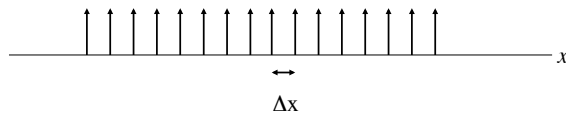


Other names: Impulse train, bed of nails, shah function.

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Scaled Comb Function

$$\begin{aligned} \text{comb}\left(\frac{x}{\Delta x}\right) &= \sum_{n=-\infty}^{\infty} \delta\left(\frac{x}{\Delta x} - n\right) \\ &= \sum_{n=-\infty}^{\infty} \delta\left(\frac{x - n\Delta x}{\Delta x}\right) \\ &= \Delta x \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x) \end{aligned}$$



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1D spatial sampling

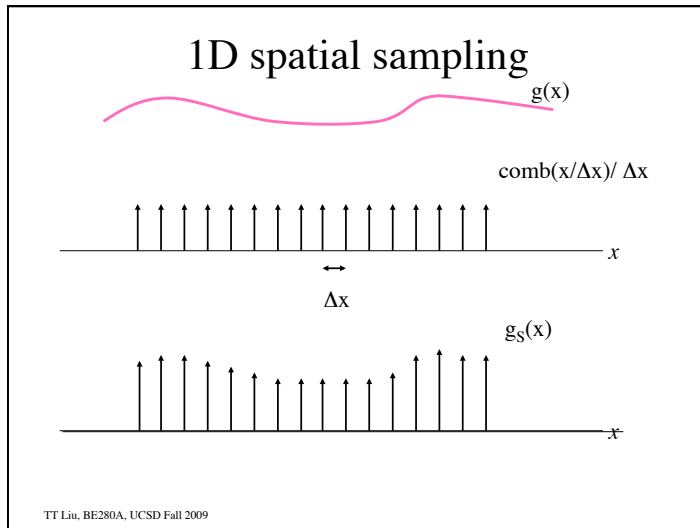
$$\begin{aligned} g_s(x) &= g(x) \frac{1}{\Delta x} \text{comb}\left(\frac{x}{\Delta x}\right) \\ &= g(x) \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x) \\ &= \sum_{n=-\infty}^{\infty} g(n\Delta x) \delta(x - n\Delta x) \end{aligned}$$

Recall the sifting property $\int_{-\infty}^{\infty} g(x) \delta(x - a) = g(a)$

But we can also write $\int_{-\infty}^{\infty} g(a) \delta(x - a) = g(a) \int_{-\infty}^{\infty} \delta(x - a) = g(a)$

So, $g(x) \delta(x - a) = g(a) \delta(x - a)$

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Fourier Transform of comb(x)

$$F[\text{comb}(x)] = \text{comb}(k_x)$$

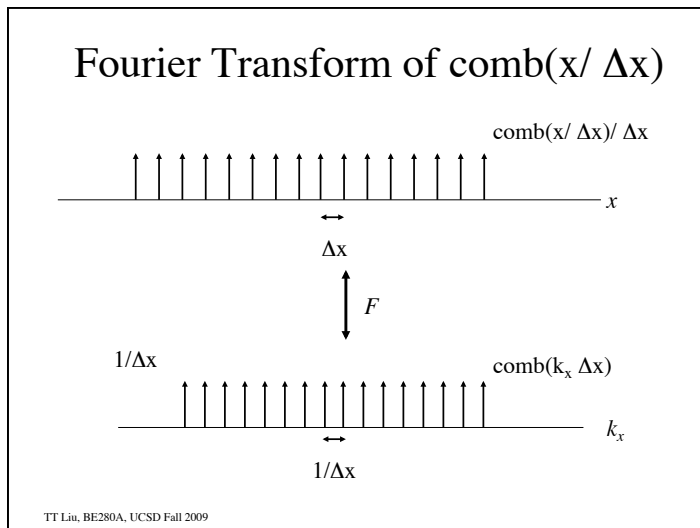
$$= \sum_{n=-\infty}^{\infty} \delta(k_x - n)$$

$$F\left[\frac{1}{\Delta x} \text{comb}\left(\frac{x}{\Delta x}\right)\right] = \frac{1}{\Delta x} \Delta x \text{comb}(k_x \Delta x)$$

$$= \sum_{n=-\infty}^{\infty} \delta(k_x \Delta x - n)$$

$$= \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} \delta\left(k_x - \frac{n}{\Delta x}\right)$$

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Fourier Transform of $g_S(x)$

$$F[g_S(x)] = F\left[g(x) \frac{1}{\Delta x} \text{comb}\left(\frac{x}{\Delta x}\right)\right]$$

$$= G(k_x) * F\left[\frac{1}{\Delta x} \text{comb}\left(\frac{x}{\Delta x}\right)\right]$$

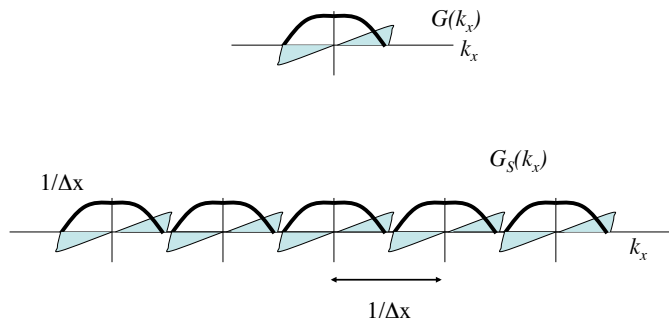
$$= G(k_x) * \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} \delta\left(k_x - \frac{n}{\Delta x}\right)$$

$$= \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} G(k_x) * \delta\left(k_x - \frac{n}{\Delta x}\right)$$

$$= \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} G\left(k_x - \frac{n}{\Delta x}\right)$$

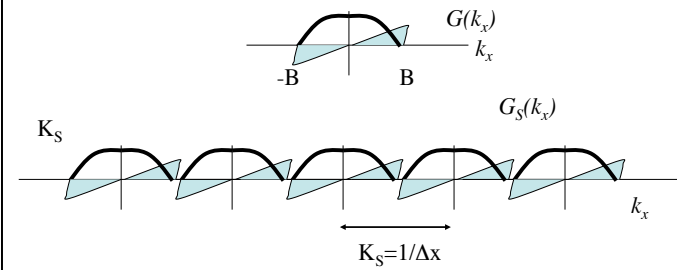
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Fourier Transform of $g_S(x)$



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Nyquist Condition



To avoid overlap, we require that $1/\Delta x > 2B$ or $K_S > 2B$ where $K_S = 1/\Delta x$ is the sampling frequency

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Example

Assume that the highest spatial frequency in an object is $B = 2 \text{ cm}^{-1}$.

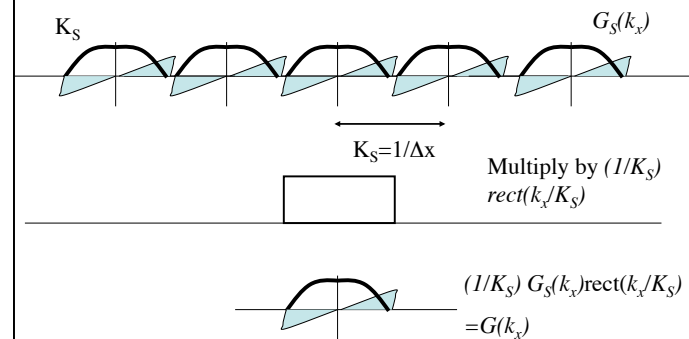
Thus, smallest spatial period is 0.5 cm .

Nyquist theorem says we need to sample with $\Delta x < 1/2B = 0.25 \text{ cm}$

This corresponds to 2 samples per spatial period.

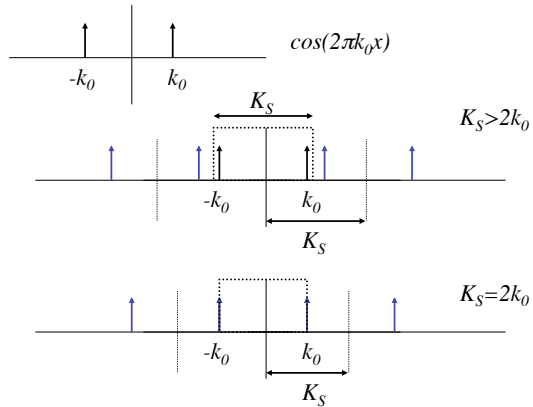
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Reconstruction from Samples



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Example Cosine Reconstruction



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Reconstruction from Samples

If the Nyquist condition is met, then

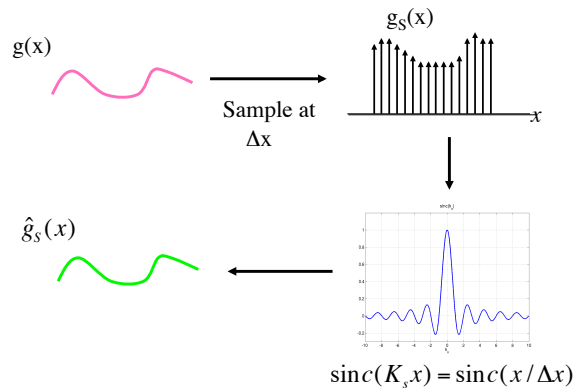
$$\hat{G}_S(k_x) = \frac{1}{K_S} G_S(k_x) \text{rect}(k_x / K_S) = G(k_x)$$

And the signal can be reconstructed by convolving the sample with a sinc function

$$\begin{aligned} \hat{g}_S(x) &= g_S(x) * \text{sinc}(K_S x) \\ &= \left(\sum_{n=-\infty}^{\infty} g(n\Delta X) \delta(x - n\Delta X) \right) * \text{sinc}(K_S x) \\ &= \sum_{n=-\infty}^{\infty} g(n\Delta X) \text{sinc}(K_S(x - n\Delta X)) \end{aligned}$$

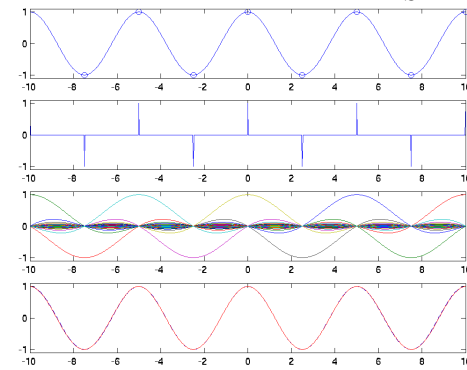
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Reconstruction from Samples



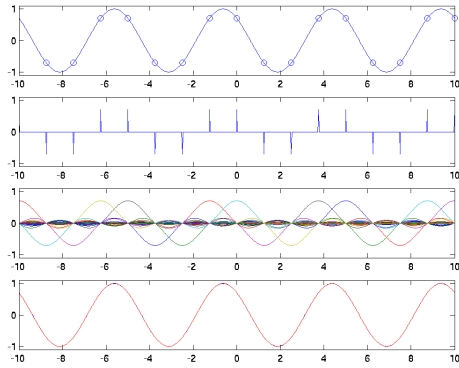
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Cosine Example with $K_S = 2k_0$



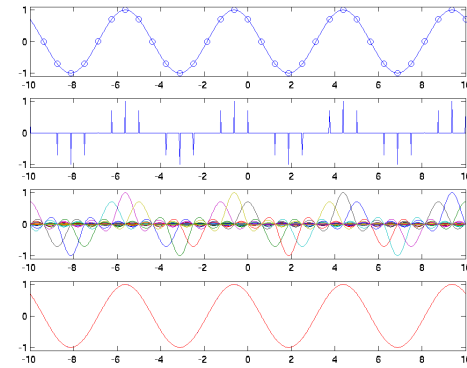
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Example with $K_s=4k_0$



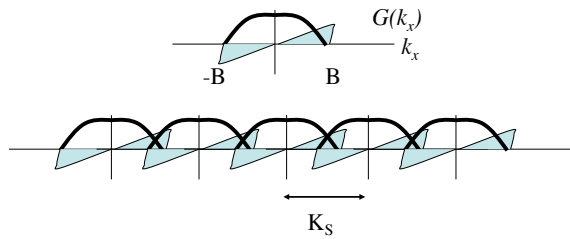
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Example with $K_s=8k_0$



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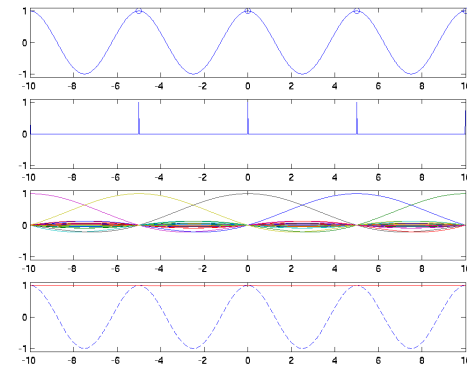
Aliasing



Aliasing occurs when the Nyquist condition is not satisfied.
This occurs for $K_s \leq 2B$

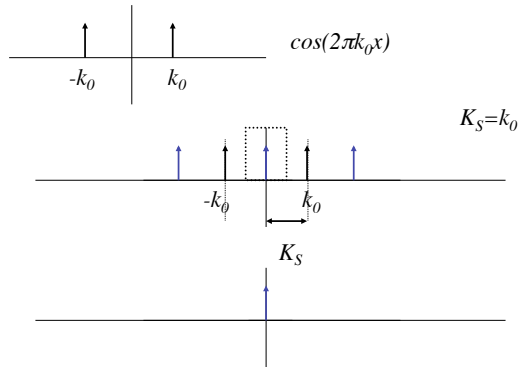
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Aliasing Example



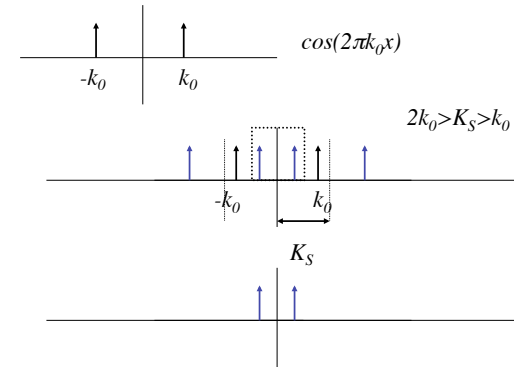
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Aliasing Example



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Aliasing Example



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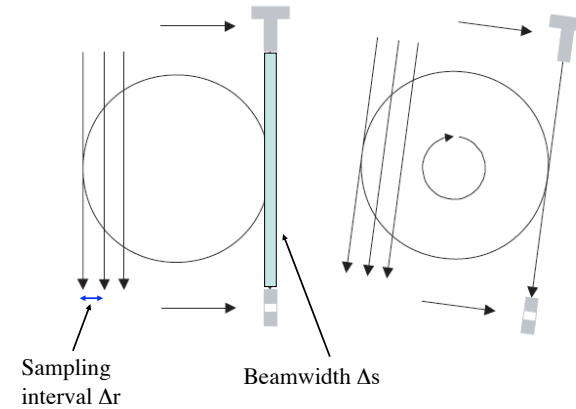
Example

1. Consider the function $g(x) = \cos^2(2\pi k_0 x)$. Sketch this function. You sample this signal in the spatial domain with a sampling rate $K_s = 1/\Delta x$ (e.g. samples spaced at intervals of Δx). What is the minimum sampling rate that you can use without aliasing? Give an intuitive explanation for your answer.

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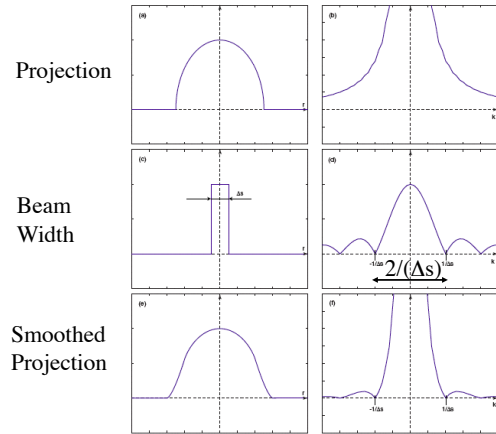
Detector Sampling Requirements



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Smoothing of Projection



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Smoothing of Projection

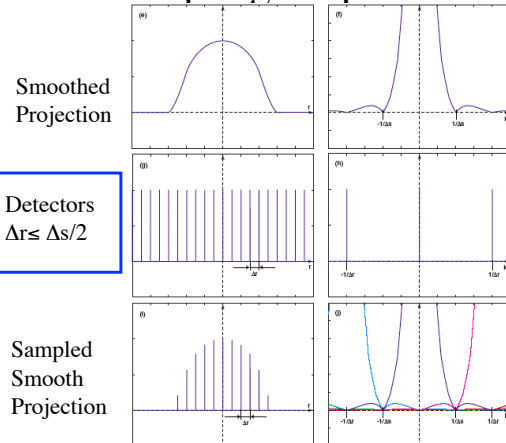
$$g_s(l, \theta) = \text{rect}(l/\Delta s) * g(l, \theta)$$

$$G_s(k_x, \theta) = \Delta s \text{sinc}(k_x \Delta s) G(k_x, \theta)$$

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Sampling Requirements

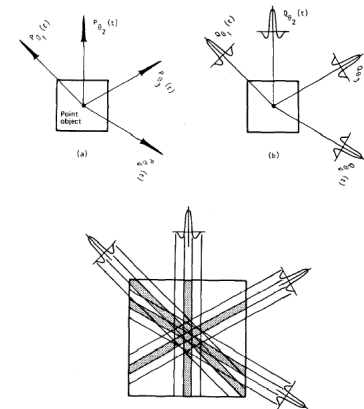


Detectors
 $\Delta r \leq \Delta s/2$

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View Aliasing



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View Sampling Requirements

View Sampling -- how many views?

Basic idea is that to make the maximum angular sampling the same as the projection sampling.

$$\frac{\pi FOV}{N_{views}} = \Delta r$$

$$N_{views,360} = \frac{\pi FOV}{\Delta r} = \pi N_{proj} \quad (\text{for 360 degrees})$$

$$N_{views,180} = \frac{\pi N_{proj}}{2} \quad (\text{for 180 degrees})$$

Example

beamwidth $\Delta s = 1$ mm

Field of View (FOV) = 50 cm

$\Delta r = \Delta s/2 = 0.5$ mm

500 mm / 0.5 mm = N = 1000 detector samples

$\pi * N = 3146$ views per 360 degrees

≈ 1500 views per 180 degrees

CT "Rule of Thumb"

$$N_{view} = N_{detectors} = N_{pixels}$$