Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2009
MRI Lecture 2a

Gradient Fields

Define

$$\vec{G} \equiv G_x\hat{i} + G_y\hat{j} + G_z\hat{k} \qquad \vec{r} \equiv x\hat{i} + y\hat{j} + z\hat{k}$$

So that

$$G_x x + G_y y + G_z z = \vec{G}\cdot\vec{r}$$

Also, let the gradient fields be a function of time. Then the z-directed magnetic field at each point in the volume is given by :

$$B_z(\vec{r},t) = B_0 + \vec{G}(t)\cdot\vec{r}$$

Static Gradient Fields

In a uniform magnetic field, the transverse magnetization is given by:

$$M(t) = M(0)e^{-j\omega_0 t}e^{-t/T_2}$$

In the presence of non time-varying gradients we have

$$\begin{aligned} M(\vec{r}) &= M(\vec{r},0)e^{-j\gamma B_z(\vec{r})t}e^{-t/T_2(\vec{r})} \\ &= M(\vec{r},0)e^{-j\gamma(B_0+\vec{G}\cdot\vec{r})t}e^{-t/T_2(\vec{r})} \\ &= M(\vec{r},0)e^{-j\omega_0 t}e^{-j\gamma\vec{G}\cdot\vec{r}t}e^{-t/T_2(\vec{r})} \end{aligned}$$

Time-Varying Gradient Fields

In the presence of time-varying gradients the frequency as a function of space and time is:

$$\begin{aligned} \omega(\vec{r},t) &= \gamma B_z(\vec{r},t) \\ &= \gamma B_0 + \gamma\vec{G}(t)\cdot\vec{r} \\ &= \omega_0 + \Delta\omega(\vec{r},t) \end{aligned}$$

Phase

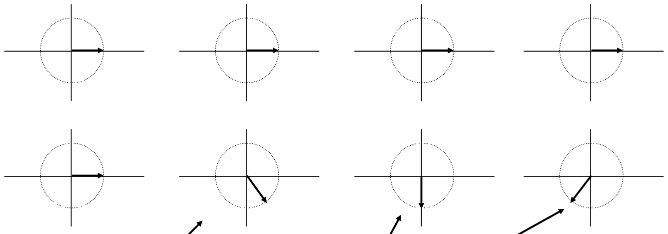
Phase = angle of the magnetization phasor
Frequency = rate of change of angle (e.g. radians/sec)
Phase = time integral of frequency

$$\begin{aligned}\varphi(\vec{r},t) &= -\int_0^t \omega(\vec{r},\tau)d\tau \\ &= -\omega_0 t + \Delta\varphi(\vec{r},t)\end{aligned}$$

Where the incremental phase due to the gradients is

$$\begin{aligned}\Delta\varphi(\vec{r},t) &= -\int_0^t \Delta\omega(\vec{r},\tau)d\tau \\ &= -\int_0^t \gamma\vec{G}(\vec{r},\tau)\cdot\vec{r}\,d\tau\end{aligned}$$

Phase with constant gradient



$$\Delta\varphi(\vec{r},t_1) = -\int_0^{t_1} \Delta\omega(\vec{r},\tau)d\tau$$

$$\begin{aligned}\Delta\varphi(\vec{r},t_2) &= -\int_0^{t_2} \Delta\omega(\vec{r},\tau)d\tau \\ &= -\Delta\omega(\vec{r})t_2 \\ &\text{if } \Delta\omega \text{ is non - time varying.}\end{aligned}$$

$$\Delta\varphi(\vec{r},t_3) = -\int_0^{t_3} \Delta\omega(\vec{r},\tau)d\tau$$

Time-Varying Gradient Fields

The transverse magnetization is then given by

$$\begin{aligned}M(\vec{r},t) &= M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{\varphi(\vec{r},t)} \\ &= M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\int_o^t \Delta\omega(\vec{r},t)d\tau\right) \\ &= M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)\end{aligned}$$

Signal Equation

Signal from a volume

$$\begin{aligned}s_r(t) &= \int_V M(\vec{r},t)dV \\ &= \int_x\int_y\int_z M(x,y,z,0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)dxdydz\end{aligned}$$

For now, consider signal from a slice along z and drop the T_2 term. Define $m(x,y) \equiv \int_{z_0-\Delta z/2}^{z_0+\Delta z/2} M(\vec{r},t)dz$

To obtain

$$s_r(t) = \int_x\int_y m(x,y)e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)dxdy$$

Signal Equation

Demodulate the signal to obtain

$$
\begin{aligned}
s(t) &= e^{j\omega_0 t} s_r(t) \\
&= \int_x \int_y m(x,y)\exp\left(-j\gamma \int_o^t \vec{G}(\tau)\cdot \vec{r}\, d\tau\right) dxdy \\
&= \int_x \int_y m(x,y)\exp\left(-j\gamma \int_o^t \left[G_x(\tau)x + G_y(\tau)y\right] d\tau\right) dxdy \\
&= \int_x \int_y m(x,y)\exp\left(-j2\pi\left(k_x(t)x + k_y(t)y\right)\right) dxdy
\end{aligned}
$$

Where

$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

$$k_y(t) = \frac{\gamma}{2\pi}\int_0^t G_y(\tau)d\tau$$

MR signal is Fourier Transform

$$
\begin{aligned}
s(t) &= \int_x \int_y m(x,y)\exp\left(-j2\pi\left(k_x(t)x + k_y(t)y\right)\right) dxdy \\
&= M\left(k_x(t), k_y(t)\right) \\
&= F\left[m(x,y)\right]\Big|_{k_x(t),k_y(t)}
\end{aligned}
$$

Recap

- Frequency = rate of change of phase.
- Higher magnetic field -> higher Larmor frequency -> phase changes more rapidly with time.
- With a constant gradient G_x, spins at different x locations precess at different frequencies -> spins at greater x-values change phase more rapidly.
- With a constant gradient, distribution of phases across x locations changes with time. (phase modulation)
- More rapid change of phase with x -> higher spatial frequency k_x

K-space

At each point in time, the received signal is the Fourier transform of the object

$$s(t) = M\left(k_x(t), k_y(t)\right) = F\left[m(x,y)\right]\Big|_{k_x(t),k_y(t)}$$

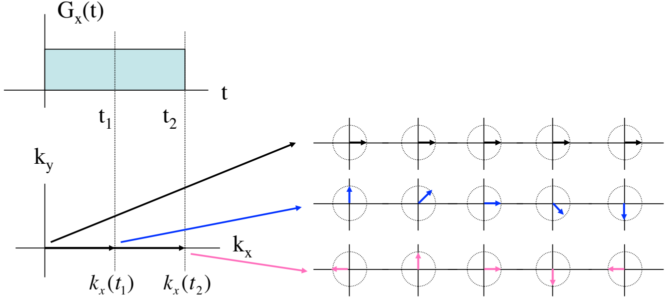
evaluated at the spatial frequencies:

$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

$$k_y(t) = \frac{\gamma}{2\pi}\int_0^t G_y(\tau)d\tau$$

Thus, the gradients control our position in k-space. The design of an MRI pulse sequence requires us to efficiently cover enough of k-space to form our image.

K-space trajectory



$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

Units

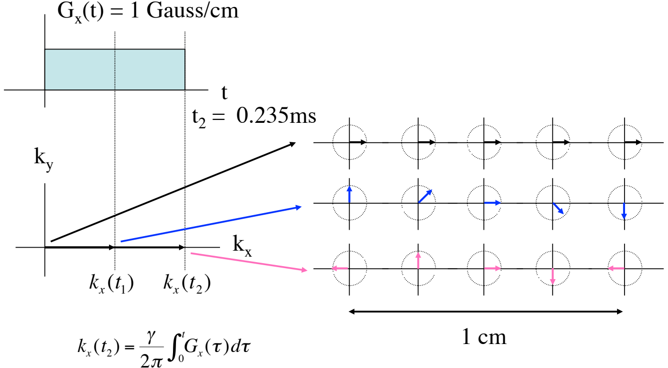
Spatial frequencies (k_x, k_y) have units of 1/distance.
Most commonly, 1/cm

Gradient strengths have units of (magnetic field)/distance. Most commonly G/cm or mT/m

γ/(2π) has units of Hz/G or Hz/Tesla.

$$\begin{aligned} k_x(t) &= \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau \\ &= [Hz/Gauss][Gauss/cm][\sec] \\ &= [1/cm] \end{aligned}$$

Example



$$\begin{aligned} k_x(t_2) &= \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau \\ &= 4257 Hz/G \cdot 1G/cm \cdot 0.235\times10^{-3} s \\ &= 1\ cm^{-1} \end{aligned}$$