Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2010
CT/Fourier Lecture 2

Topics

- Projection Slice Theorem
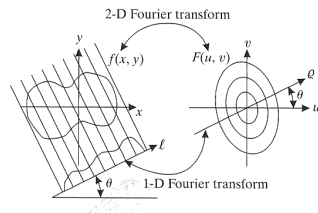- Fourier Transforms

Projection Slice Theorem

$$G(\rho,\theta) = \int_{-\infty}^{\infty} g(l,\theta)e^{-j2\pi\rho l}dl$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - l)e^{-j2\pi\rho l}dx\ dy\ dl$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi\rho(x\cos\theta + y\sin\theta)}dx\ dy$$

$$= F_{2D}\big[f(x,y)\big]\big|_{u=\rho\cos\theta,\,v=\rho\sin\theta}$$

2-D Fourier transform

1-D Fourier transform
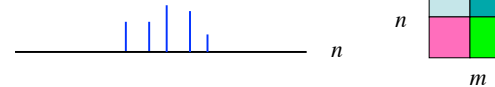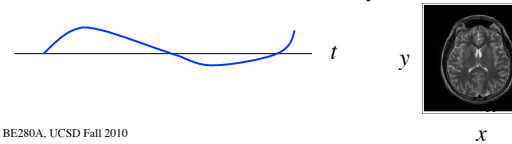
Prince&Links 2006

Signals and Images

Discrete-time/space signal/image: continuous valued function with a discrete time/space index, denoted as $s[n]$ for 1D, $s[m,n]$ for 2D , etc.
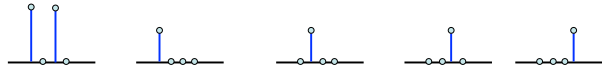

Continuous-time/space signal/image: continuous valued function with a continuous time/space index, denoted as $s(t)$ or $s(x)$ for 1D, $s(x,y)$ for 2D, etc.
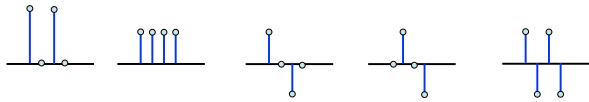
1

1D Signal Decomposition

$$\{2,0,2,0\} = 2 \cdot \{1,0,0,0\} + 0 \cdot \{0,1,0,0\} + 2 \cdot \{0,0,1,0\} + 0 \cdot \{0,0,0,1\}$$

$$\{2,0,2,0\} = a \cdot \{1,1,1,1\} + b \cdot \{1,0,-1,0\} + c \cdot \{0,1,0,-1\} + d \cdot \{1,-1,1,-1\}$$
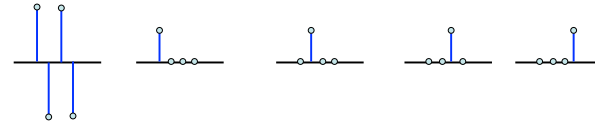
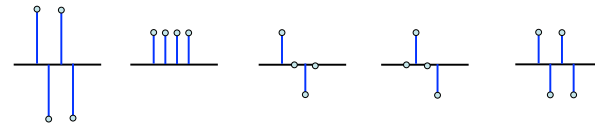$$\{2,0,2,0\} = 1 \cdot \{1,1,1,1\} + 0 \cdot \{1,0,-1,0\} + 0 \cdot \{0,1,0,-1\} + 1 \cdot \{1,-1,1,-1\}$$

1D Signal Decomposition

$$\{2,-2,2,-2\} = 2 \cdot \{1,0,0,0\} - 2 \cdot \{0,1,0,0\} + 2 \cdot \{0,0,1,0\} - 2 \cdot \{0,0,0,1\}$$

$$\{2,-2,2,-2\} = 0 \cdot \{1,1,1,1\} + 0 \cdot \{1,0,-1,0\} + 0 \cdot \{0,1,0,-1\} + 2 \cdot \{1,-1,1,-1\}$$

Eskimo Words for Snow

```
tlapa          powder snow
tlacringit     snow that is crusted on the surface
kayi           drifting snow
tlapat         still snow
klin           remembered snow
naklin         forgotten snow
tlamo          snow that falls in large wet flakes
tlatim         snow that falls in small flakes
tlaslo         snow that falls slowly
tlapinti       snow that falls quickly
kripya         snow that has melted and refrozen
tliyel         snow that has been marked by wolves
tliyelin       snow that has been marked by Eskimos

tlalman        snow sold to German tourists
tlalam         snow sold to American tourists
tlanip         snow sold to Japanese tourists

tla-na-na      snow mixed with the sound of old rock and roll
                 from a portable radio
depptla        a small snowball, preserved in Lucite, that had been handled
                 by Johnny Depp


                       http://www.mendosa.com/snow.html
```

Image Compression

2

2D Image

a	b
c	d

$=$

a	0
0	0

$+$

0	b
0	0

$+$

0	0
c	0

$+$

0	0
0	d

Image Decomposition

a	b
c	d

$=$

a

1	0
0	0

$+ b$

0	1
0	0

$+$

c

0	0
1	0

$+ d$

0	0
0	1

Basis Functions   Coefficients
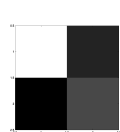
x   =

1/2	1/2
1/2	1/2

1/2	-1/2
1/2	-1/2

1/2	1/2
-1/2	-1/2

1/2	-1/2
-1/2	1/2

Sum

Object

1D Fourier Transform

KPBS

KIFM

KIOZ

Fourier

Transform

3
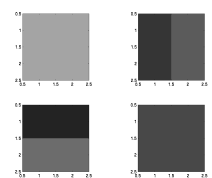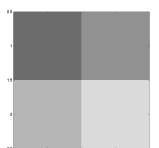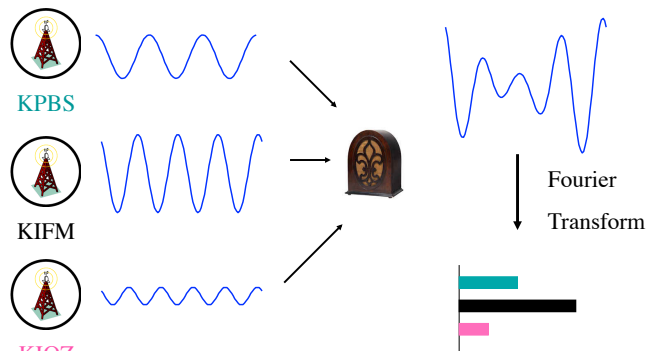
The Fourier Transform

Fourier Transform (FT)

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt = F\{g(t)\}$$

Inverse Fourier Transform

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df = F^{-1}\{G(f)\}$$

Complex Numbers

$$j = \sqrt{-1}$$

$$j^2 = ?$$
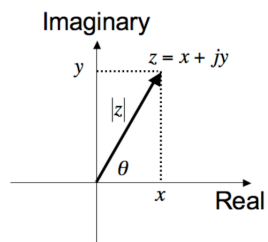$$(3 + 2j)(3 - 2j) = ?$$

$$j^2 = -1$$
$$(3 + 2j)(3 - 2j) = 9 - 4j^2 = 13$$

Complex Numbers

$$z = 2 + 1j$$

$$|z| = \sqrt{2^2 + 1} = \sqrt{5}$$
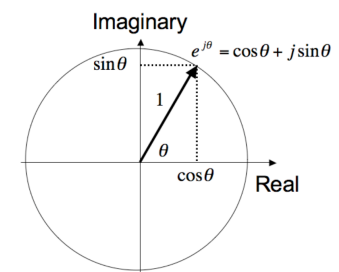
$$\theta = \tan^{-1}\left(\frac{1}{2}\right) = 30 \text{ degrees}$$

Euler's Formula
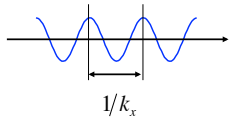
$$e^{j\theta} = \cos\theta + j\sin\theta$$
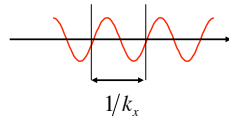
$$z = x + jy = |z|e^{j\theta}$$

4

1D Fourier Transform

$$G(k_x) = \int_{-\infty}^{\infty} g(x)\exp(-j2\pi k_x x)dx$$

$$= \int_{-\infty}^{\infty} g(x)\cos(2\pi k_x x)dx - j\int_{-\infty}^{\infty} g(x)\sin(2\pi k_x x)dx$$

The part of $g(x)$ that "looks"
like $\cos(2\pi k_x x)$

The part of $g(x)$ that "looks"
like $\sin(2\pi k_x x)$

$1/k_x$

$1/k_x$

Units

Temporal Coordinates, e.g. t in seconds, f in cycles/second

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt \qquad \text{Fourier Transform}$$

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df \qquad \text{Inverse Fourier Transform}$$

Spatial Coordinates, e.g. x in cm, k_x is spatial frequency in cycles/cm

$$G(k_x) = \int_{-\infty}^{\infty} g(x)e^{-j2\pi k_x x}dx \qquad \text{Fourier Transform}$$

$$g(x) = \int_{-\infty}^{\infty} G(k_x)e^{j2\pi k_x x}dk_x \qquad \text{Inverse Fourier Transform}$$

2D Fourier Transform

Fourier Transform

$$G(k_x, k_y) = F\big[g(x,y)\big] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi\left(k_x x + k_y y\right)}dxdy$$
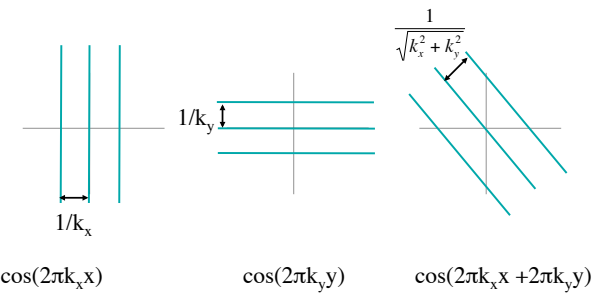
Inverse Fourier Transform

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x, k_y)e^{j2\pi\left(k_x x + k_y y\right)}dk_x dk_y$$
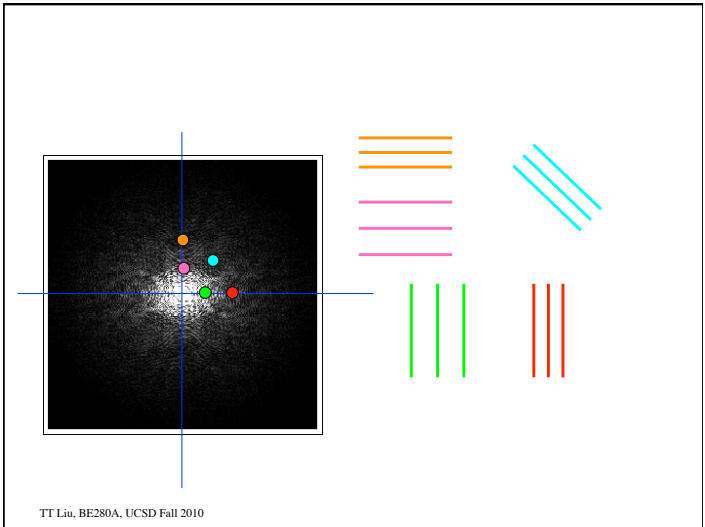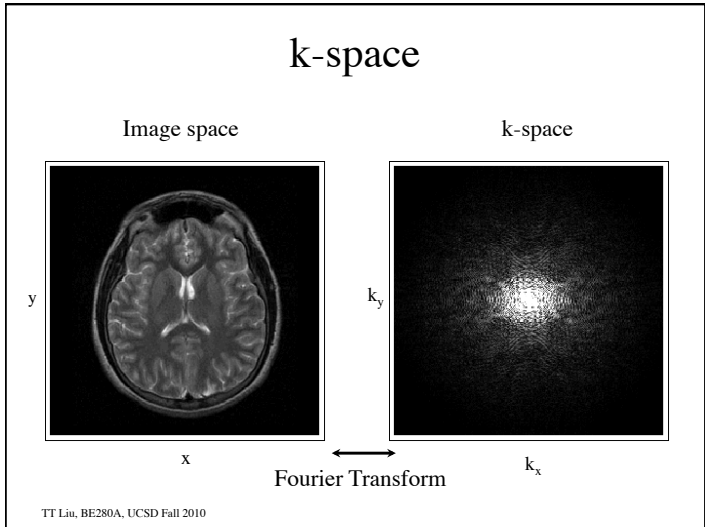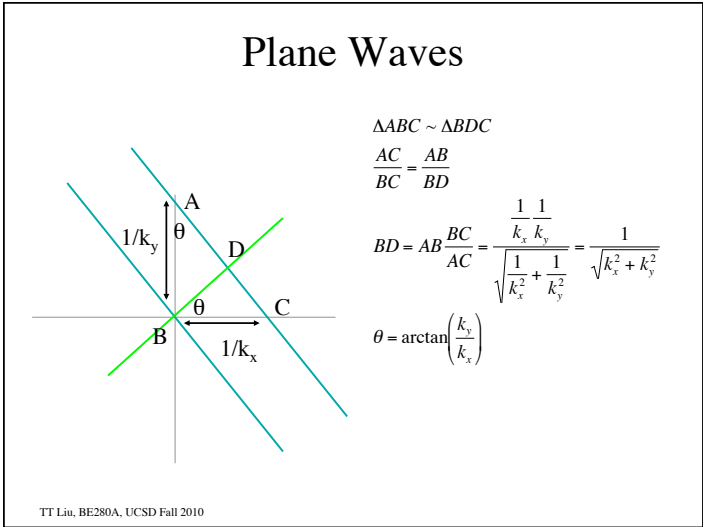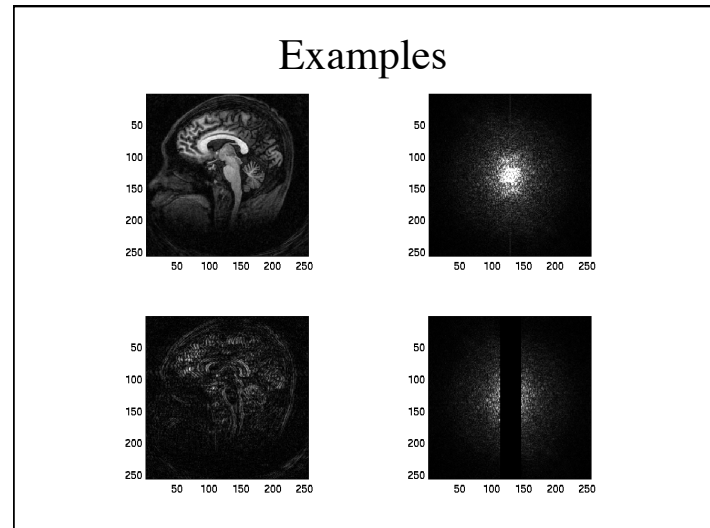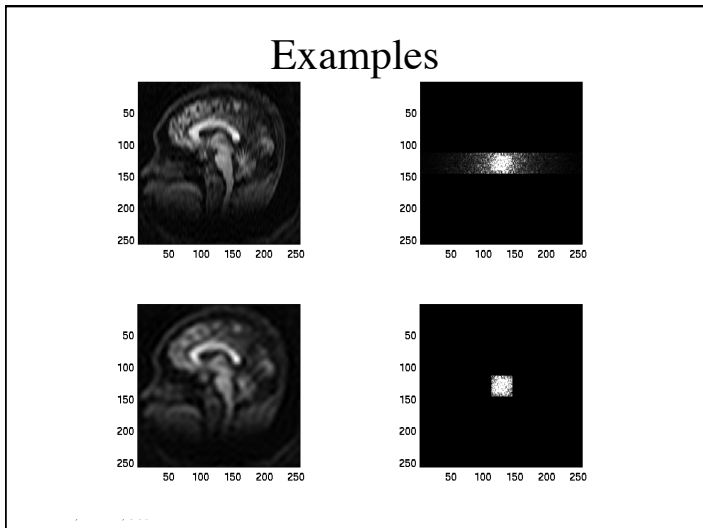
Plane Waves

$$e^{j2\pi(k_x x + k_y y)} = \cos\big(2\pi(k_x x + k_y y)\big) + j\sin\big(2\pi(k_x x + k_y y)\big)$$

$\dfrac{1}{\sqrt{k_x^2 + k_y^2}}$

$1/k_y$

$1/k_x$

$\cos(2\pi k_x x)$ $\qquad$ $\cos(2\pi k_y y)$ $\qquad$ $\cos(2\pi k_x x + 2\pi k_y y)$

Figure 2.5 from Prince and Link

Plane Waves

$$\Delta ABC \sim \Delta BDC$$

$$\frac{AC}{BC} = \frac{AB}{BD}$$

$$BD = AB\frac{BC}{AC} = \frac{\frac{1}{k_x}\frac{1}{k_y}}{\sqrt{\frac{1}{k_x^2} + \frac{1}{k_y^2}}} = \frac{1}{\sqrt{k_x^2 + k_y^2}}$$

$$\theta = \arctan\left(\frac{k_y}{k_x}\right)$$

k-space

Image space

k-space

Fourier Transform

Examples

Examples

Examples

Examples

Examples