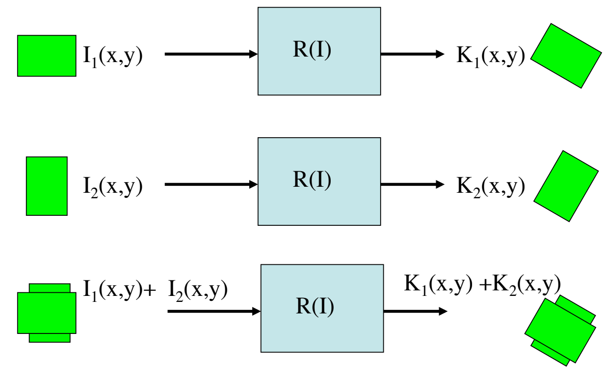
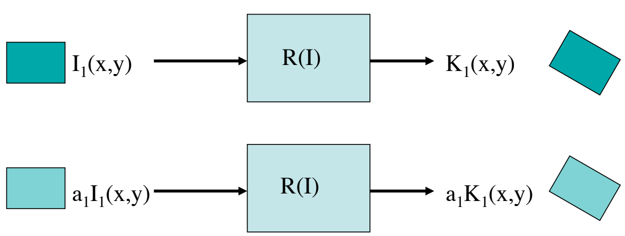
Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2010
X-Rays Lecture 2

Linearity (Addition)



Linearity (Scaling)



Linearity

A system R is linear if for two inputs $I_1(x,y)$ and $I_2(x,y)$ with outputs

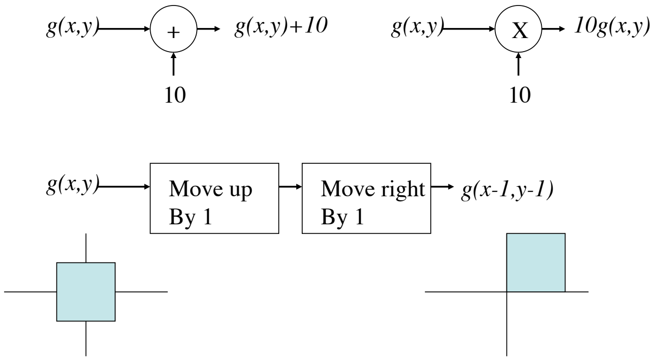
$R(I_1(x,y))=K_1(x,y)$ and $R(I_2(x,y))=K_2(x,y)$

the response to the weighted sum of inputs is the weighted sum of outputs:

$$R(a_1I_1(x,y)+ a_2I_2(x,y))=a_1K_1(x,y)+ a_2K_2(x,y)$$

Example

Are these linear systems?



Superposition

$$g[m] = g[0]\delta[m] + g[1]\delta[m-1] + g[2]\delta[m-2]$$
$$h[m',k] = L[\delta[m-k]]$$

$$\begin{aligned}
y[m'] &= L\big[g[m]\big] \\
&= L\big[g[0]\delta[m] + g[1]\delta[m-1] + g[2]\delta[m-2]\big] \\
&= L\big[g[0]\delta[m]\big] + L\big[g[1]\delta[m-1]\big] + L\big[g[2]\delta[m-2]\big] \\
&= g[0]L\big[\delta[m]\big] + g[1]L\big[\delta[m-1]\big] + g[2]L\big[\delta[m-2]\big] \\
&= g[0]h[m',0] + g[1]h[m',1] + g[2]h[m',2] \\
&= \sum_{k=0}^{2} g[k]h[m',k]
\end{aligned}$$

Superposition Integral

What is the response to an arbitrary function $g(x_1,y_1)$?

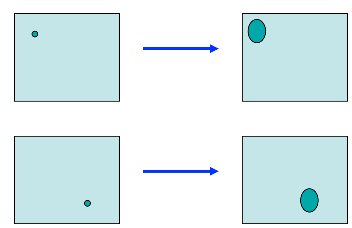
Write $g(x_1,y_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x_1-\xi,y_1-\eta)\,d\xi d\eta.$

The response is given by

$$\begin{aligned}
I(x_2,y_2) &= L\big[g_1(x_1,y_1)\big] \\
&= L\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x_1-\xi,y_1-\eta)\,d\xi d\eta\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)L\big[\delta(x_1-\xi,y_1-\eta)\big]d\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2,y_2;\xi,\eta)\,d\xi d\eta
\end{aligned}$$

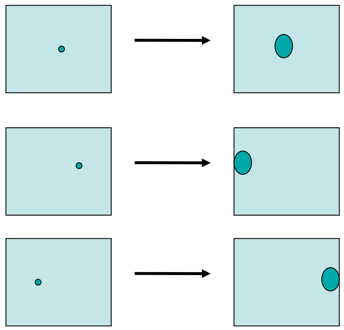
Space Invariance

If a system is space invariant, the impulse response depends only on the difference between the output coordinates and the position of the impulse and is given by $h(x_2,y_2;\xi,\eta) = h(x_2-\xi,y_2-\eta)$



Pinhole Magnification Example

____, the pinhole system ____ space invariant.



Convolution

$$g[m] = g[0]\delta[m] + g[1]\delta[m-1] + g[2]\delta[m-2]$$
$$h[m',k] = L[\delta[m-k]] = h[m'-k]$$

$$\begin{aligned} y[m'] &= L\big[g[m]\big] \\ &= L\big[g[0]\delta[m] + g[1]\delta[m-1] + g[2]\delta[m-2]\big] \\ &= L\big[g[0]\delta[m]\big] + L\big[g[1]\delta[m-1]\big] + L\big[g[2]\delta[m-2]\big] \\ &= g[0]L\big[\delta[m]\big] + g[1]L\big[\delta[m-1]\big] + g[2]L\big[\delta[m-2]\big] \\ &= g[0]h[m'-0] + g[1]h[m'-1] + g[2]h[m'-2] \\ &= \sum_{k=0}^{2} g[k]h[m'-k] \end{aligned}$$

1D Convolution

$$\begin{aligned} I(x) &= \int_{-\infty}^{\infty} g(\xi)h(x;\xi)d\xi \\ &= \int_{-\infty}^{\infty} g(\xi)h(x-\xi)d\xi \\ &= g(x) * h(x) \end{aligned}$$

Useful fact:

$$\begin{aligned} g(x) * \delta(x-\Delta) &= \int_{-\infty}^{\infty} g(\xi)\delta(x-\Delta-\xi)d\xi \\ &= g(x-\Delta) \end{aligned}$$

2D Convolution

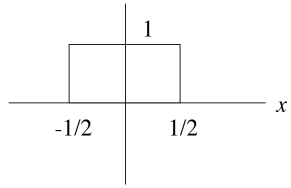
For a space invariant linear system, the superposition integral becomes a convolution integral.

$$\begin{aligned} I(x_2,y_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2,y_2;\xi,\eta)d\xi d\eta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2-\xi,y_2-\eta)d\xi d\eta \\ &= g(x_2,y_2) ** h(x_2,y_2) \end{aligned}$$

where ** denotes 2D convolution. This will sometimes be abbreviated as *, e.g. $I(x_2, y_2)= g(x_2, y_2)*h(x_2, y_2)$.

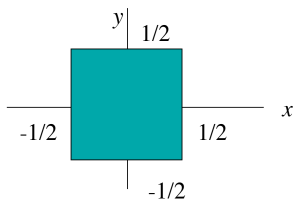
Rectangle Function

$$\Pi(x) = \begin{cases} 0 & |x| > 1/2 \\ 1 & |x| \le 1/2 \end{cases}$$

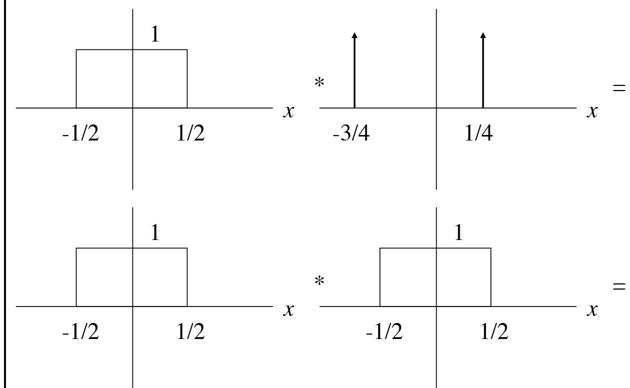


Also called rect(x)

$$\Pi(x,y) = \Pi(x)\Pi(y)$$

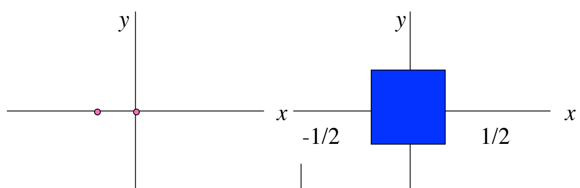


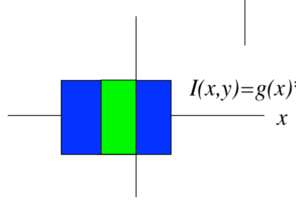
1D Convolution Examples



2D Convolution Example

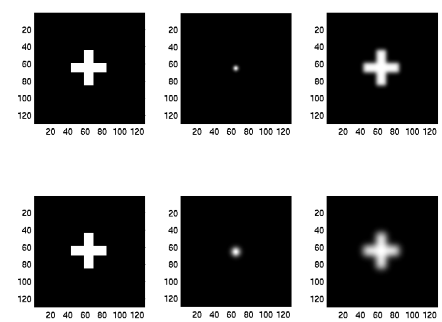
$g(x)= \delta(x+1/2,y) + \delta(x,y)$ $h(x)=rect(x,y)$



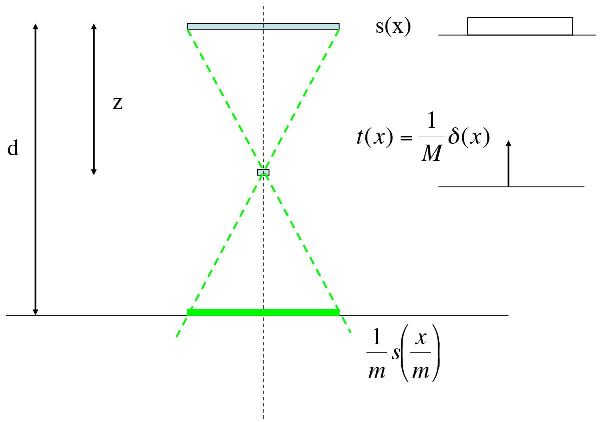


$I(x,y)=g(x)**h(x,y)$

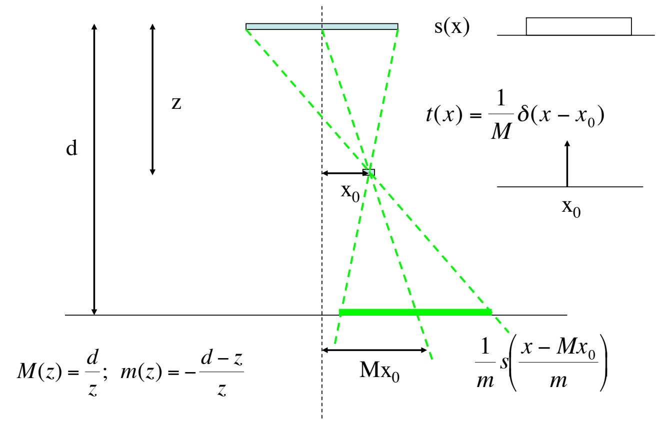
2D Convolution Example



X-Ray Imaging



X-Ray Imaging



X-Ray Imaging

For off-center pinhole object, the shifted source image can be written as

$$s\left(\frac{x - Mx_0}{m}\right) = s\left(\frac{x}{m}\right) * \frac{1}{M}\delta\left(\frac{x - Mx_0}{M}\right)$$

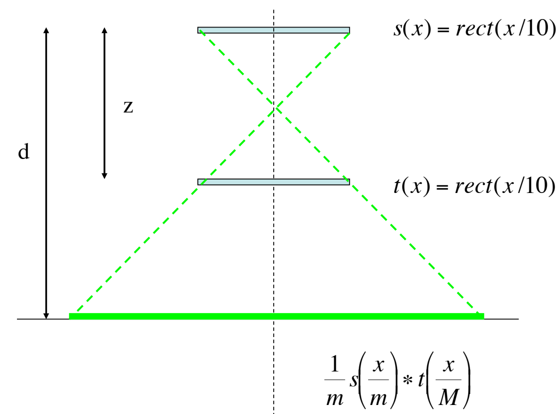
$$= s(x/m) * t\left(\frac{x}{M}\right)$$

For the general 2D case, we convolve the magnified object with the impulse response

$$I(x,y) = t\left(\frac{x}{M}, \frac{y}{M}\right) ** \frac{1}{m^2} s\left(\frac{x}{m}, \frac{y}{m}\right)$$

Note: we have ignored obliquity factors etc.

X-Ray Imaging



X-Ray Imaging

$m = 1; M = 2$

$$\frac{1}{m} s\left(\frac{x}{m}\right) * t\left(\frac{x}{M}\right) = rect(x/10) * rect(x/20)$$

$$= ???$$

Summary

1. The response to a linear system can be characterized by a spatially varying impulse response and the application of the superposition integral.
2. A shift invariant linear system can be characterized by its impulse response and the application of a convolution integral.