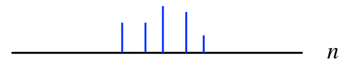
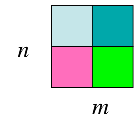
Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2012
Signals and Fourier Transforms

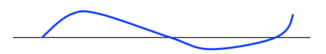
Signals and Images

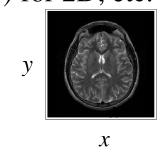
Discrete-time/space signal/image: continuous valued function with a discrete time/space index, denoted as *s[n]* for 1D, *s[m,n]* for 2D , etc.





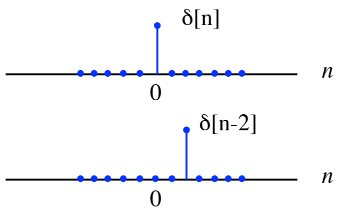
Continuous-time/space signal/image: continuous valued function with a continuous time/space index, denoted as *s(t)* or *s(x)* for 1D, *s(x,y)* for 2D, etc.





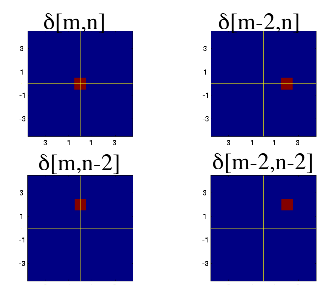
Kronecker Delta Function

$$\delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \textit{otherwise} \end{cases}$$



Kronecker Delta Function

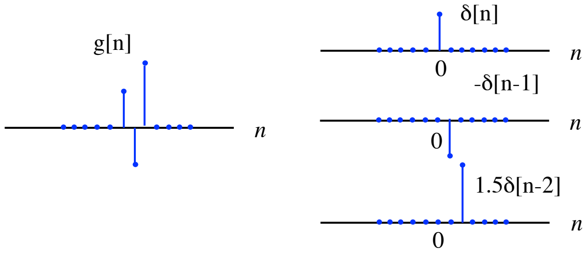
$$\delta[m,n] = \begin{cases} 1 & \text{for } m = 0, n = 0 \\ 0 & \textit{otherwise} \end{cases}$$



Discrete Signal Expansion

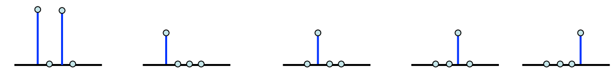
$$g[n] = \sum_{k=-\infty}^{\infty} g[k]\delta[n-k]$$

$$g[m,n] = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} g[k,l]\delta[m-k,n-l]$$

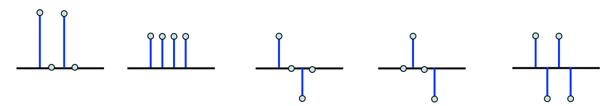


1D Signal Decomposition

$$\{2,0,2,0\} = 2\cdot\{1,0,0,0\} + 0\cdot\{0,1,0,0\} + 2\cdot\{0,0,1,0\} + 0\cdot\{0,0,0,1\}$$



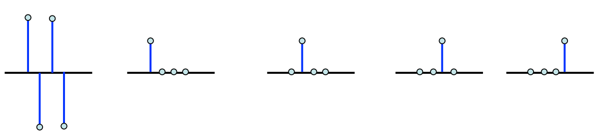
$$\{2,0,2,0\} = a\cdot\{1,1,1,1\} + b\cdot\{1,0,-1,0\} + c\cdot\{0,1,0,-1\} + d\cdot\{1,-1,1,-1\}$$



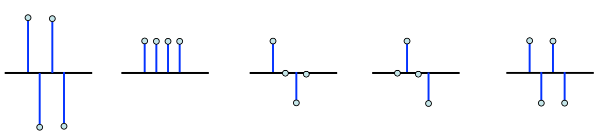
$$\{2,0,2,0\} = 1\cdot\{1,1,1,1\} + 0\cdot\{1,0,-1,0\} + 0\cdot\{0,1,0,-1\} + 1\cdot\{1,-1,1,-1\}$$

1D Signal Decomposition

$$\{2,-2,2,-2\} = 2\cdot\{1,0,0,0\} - 2\cdot\{0,1,0,0\} + 2\cdot\{0,0,1,0\} - 2\cdot\{0,0,0,1\}$$



$$\{2,-2,2,-2\} = 0\cdot\{1,1,1,1\} + 0\cdot\{1,0,-1,0\} + 0\cdot\{0,1,0,-1\} + 2\cdot\{1,-1,1,-1\}$$



2D Signal

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$$

Image Decomposition

$$\begin{bmatrix} c & d \\ a & b \end{bmatrix} = c\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$g[m,n] = a\delta[m,n] + b\delta[m,n-1] + c\delta[m-1,n] + d\delta[m-1,n-1]$$
$$= \sum_{k=0}^{1}\sum_{l=0}^{1} g[k,l]\delta[m-k,n-l]$$

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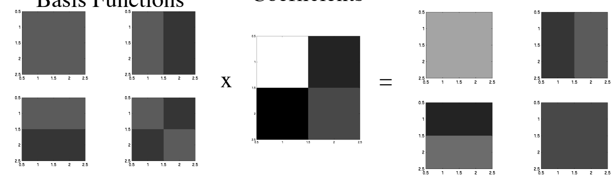

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \quad \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & -1/2 \end{bmatrix} \quad \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

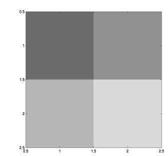

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Image Compression

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Eskimo Words for Snow

tlapa	powder snow
tlacringit	snow that is crusted on the surface
kayi	drifting snow
tlapat	still snow
klin	remembered snow
naklin	forgotten snow
tlamo	snow that falls in large wet flakes
tlatim	snow that falls in small flakes
tlaslo	snow that falls slowly
tlapinti	snow that falls quickly
kripya	snow that has melted and refrozen
tliyel	snow that has been marked by wolves
tliyelin	snow that has been marked by Eskimos
tlalman	snow sold to German tourists
tlalam	snow sold to American tourists
tlanip	snow sold to Japanese tourists
tla-na-na	snow mixed with the sound of old rock and roll from a portable radio
depptla	a small snowball, preserved in Lucite, that had been handled by Johnny Depp

http://www.mendosa.com/snow.html

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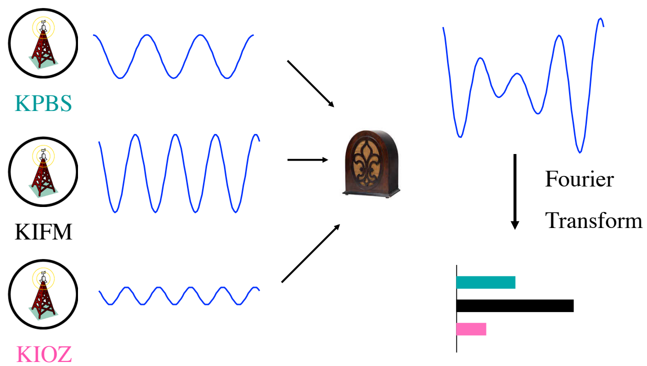
1D Fourier Transform



The Fourier Transform

Fourier Transform (FT)

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt = F\{g(t)\}$$

Inverse Fourier Transform

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df = F^{-1}\{G(f)\}$$

Complex Numbers

$j = \sqrt{-1}$

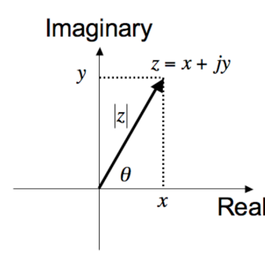
$j^2 = ?$

$(3+2j)(3-2j) = ?$

$j^2 = -1$

$(3+2j)(3-2j) = 9 - 4j^2 = 13$

Complex Numbers



$z = 2 + 1j$

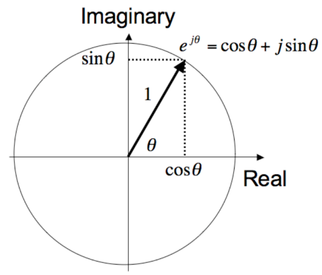
$|z| = \sqrt{2^2 + 1} = \sqrt{5}$

$\theta = \tan^{-1}\left(\frac{1}{2}\right) = 30$ degrees

Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$z = x + jy = |z|e^{j\theta}$$



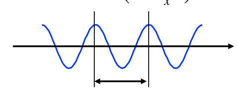
1D Fourier Transform

$$G(k_x) = \int_{-\infty}^{\infty} g(x)\exp(-j2\pi k_x x)dx$$

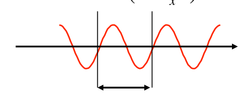
$$= \int_{-\infty}^{\infty} g(x)\cos(2\pi k_x x)dx - j\int_{-\infty}^{\infty} g(x)\sin(2\pi k_x x)dx$$

The part of $g(x)$ that "looks" like $\cos(2\pi k_x x)$

The part of $g(x)$ that "looks" like $\sin(2\pi k_x x)$



$1/k_x$



$1/k_x$

Units

Temporal Coordinates, e.g. t in seconds, f in cycles/second

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt \qquad \text{Fourier Transform}$$

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df \qquad \text{Inverse Fourier Transform}$$

Spatial Coordinates, e.g. x in cm, k_x is spatial frequency in cycles/cm

$$G(k_x) = \int_{-\infty}^{\infty} g(x)e^{-j2\pi k_x x}dx \qquad \text{Fourier Transform}$$

$$g(x) = \int_{-\infty}^{\infty} G(k_x)e^{j2\pi k_x x}dk_x \qquad \text{Inverse Fourier Transform}$$

2D Fourier Transform

Fourier Transform

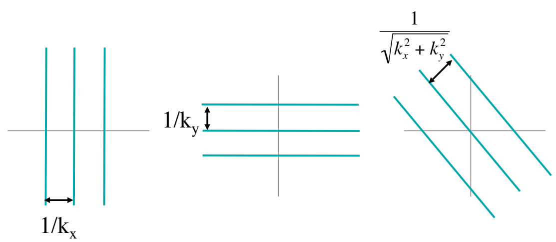
$$G(k_x,k_y) = F[g(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi(k_x x + k_y y)}dxdy$$

Inverse Fourier Transform

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x,k_y)e^{j2\pi(k_x x + k_y y)}dk_x dk_y$$

Plane Waves

$$e^{j2\pi(k_x x + k_y y)} = \cos\left(2\pi(k_x x + k_y y)\right) + j\sin\left(2\pi(k_x x + k_y y)\right)$$



$\cos(2\pi k_x x)$ $\cos(2\pi k_y y)$ $\cos(2\pi k_x x + 2\pi k_y y)$

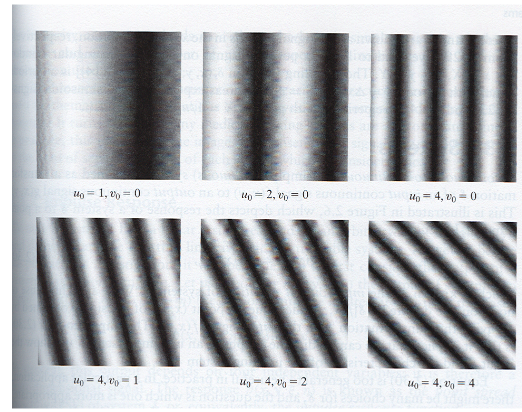
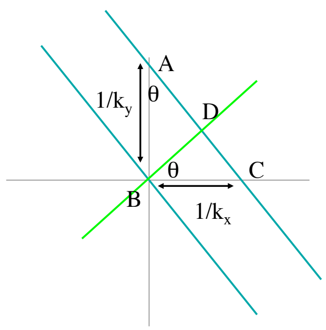


Figure 2.5 from Prince and Link

Plane Waves



$$\Delta ABC \sim \Delta BDC$$

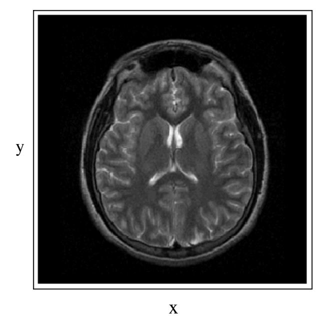
$$\frac{AC}{BC} = \frac{AB}{BD}$$

$$BD = AB\frac{BC}{AC} = \frac{\frac{1}{k_x}\frac{1}{k_y}}{\sqrt{\frac{1}{k_x^2} + \frac{1}{k_y^2}}} = \frac{1}{\sqrt{k_x^2 + k_y^2}}$$

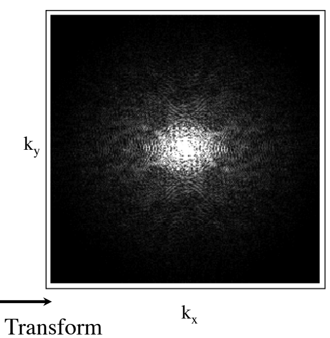
$$\theta = \arctan\left(\frac{k_y}{k_x}\right)$$

k-space

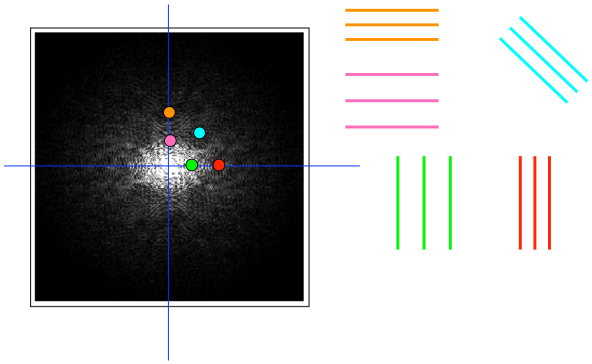
Image space

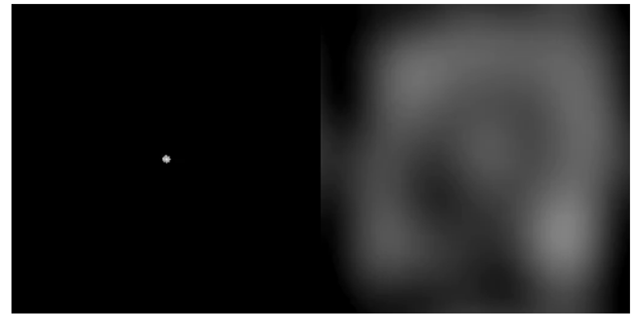


k-space

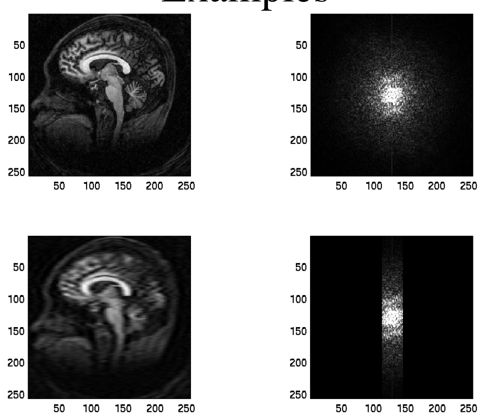


↔ Fourier Transform

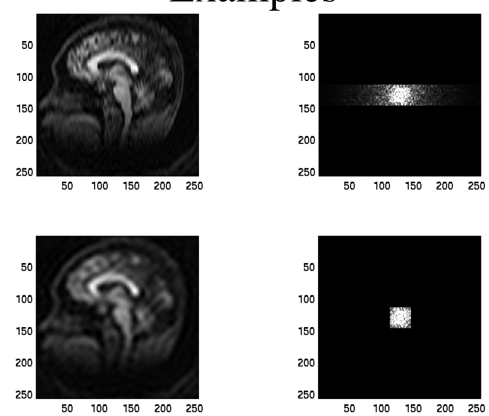




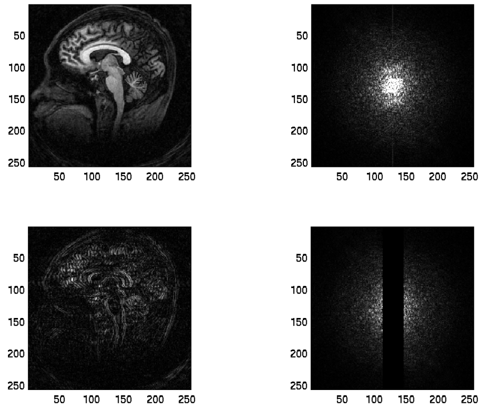
Examples



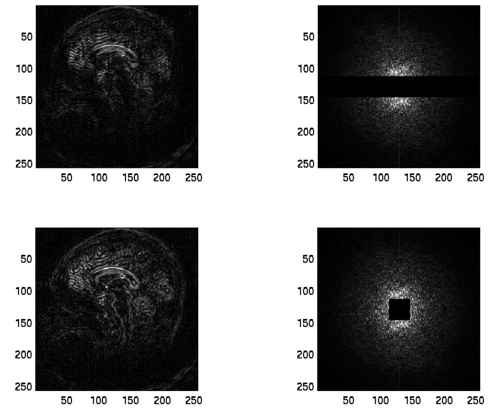
Examples



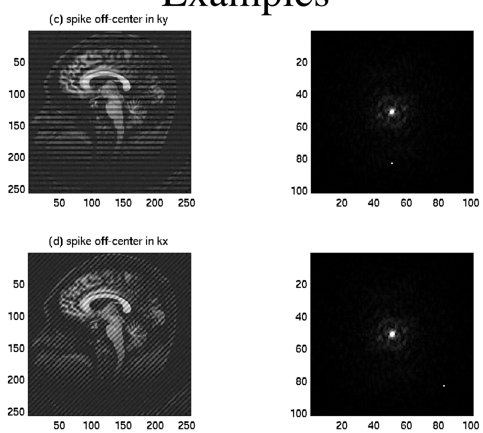
Examples

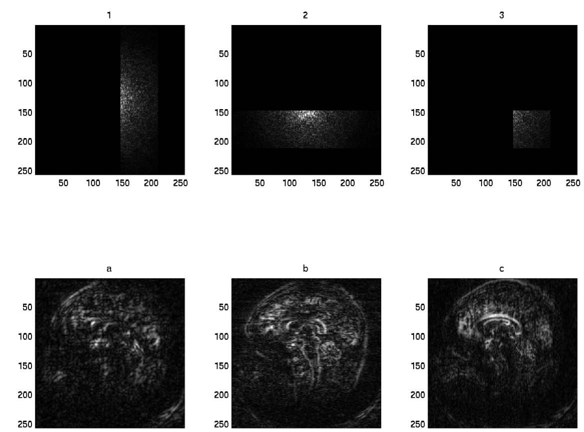


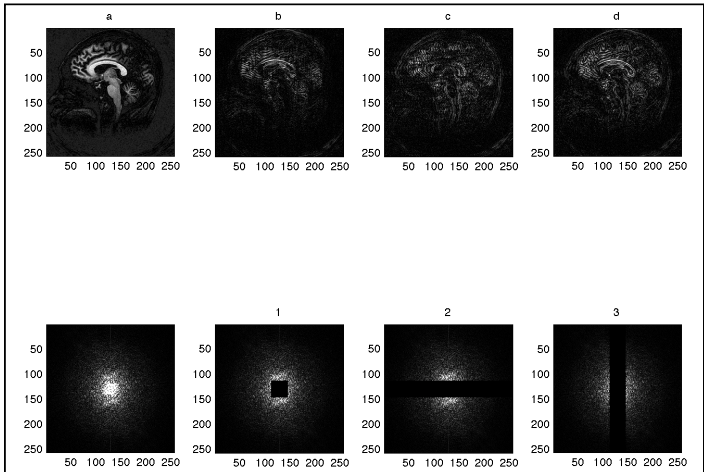
Examples

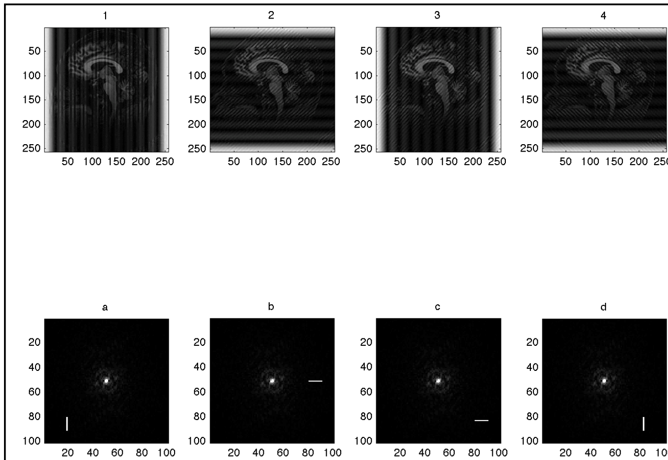


Examples









Dirac Delta Function

Notation:

$\delta(x)$ - 1D Dirac Delta Function

$\delta(x,y)$ or ${}^2\delta(x,y)$ - 2D Dirac Delta Function

$\delta(x,y,z)$ or ${}^3\delta(x,y,z)$ - 3D Dirac Delta Function

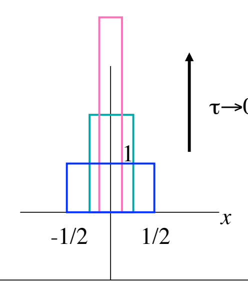
$\delta(\vec{r})$ - N Dimensional Dirac Delta Function

1D Dirac Delta Function

$\delta(x) \;=\; 0$ when $x \neq 0$ and $\int_{-\infty}^{\infty} \delta(x)dx = 1$

Can interpret the integral as a limit of the integral of an ordinary function that is shrinking in width and growing in height, while maintaining a constant area. For example, we can use a shrinking rectangle function

such that $\int_{-\infty}^{\infty} \delta(x)dx = \lim_{\tau \to 0} \int_{-\infty}^{\infty} \tau^{-1}\Pi(x/\tau)dx$.



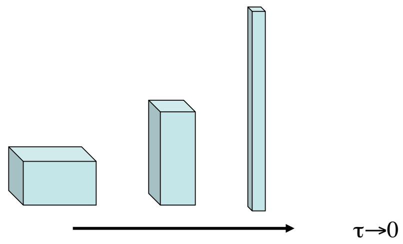
2D Dirac Delta Function

$\delta(x,y) = 0$ when $x^2 + y^2 \neq 0$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x,y)dxdy = 1$

where we can consider the limit of the integral of an ordinary 2D function that is shrinking in width but increasing in height while maintaining constant area.

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x,y)dxdy = \lim_{\tau\to 0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tau^{-2}\Pi\left(x/\tau, y/\tau\right)dxdy.$

Useful fact : $\delta(x,y) = \delta(x)\delta(y)$



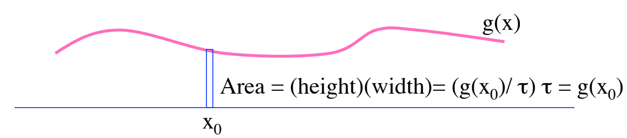
Generalized Functions

Dirac delta functions are not ordinary functions that are defined by their value at each point. Instead, they are generalized functions that are defined by what they do underneath an integral.

The most important property of the Dirac delta is the sifting property

$\int_{-\infty}^{\infty}\delta(x-x_0)g(x)dx = g(x_0)$ where $g(x)$ is a smooth function. This sifting property can be understood by considering the limiting case

$\lim_{\tau\to 0}\int_{-\infty}^{\infty}\tau^{-1}\Pi\left(x/\tau\right)g(x)dx = g(x_0)$

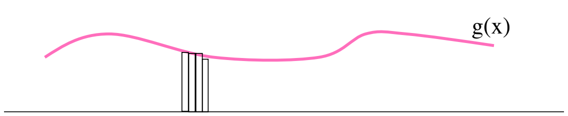


Representation of 1D Function

From the sifting property, we can write a 1D function as

$g(x) = \int_{-\infty}^{\infty}g(\xi)\delta(x-\xi)d\xi.$ To gain intuition, consider the approximation

$g(x) \approx \sum_{n=-\infty}^{\infty}g(n\Delta x)\frac{1}{\Delta x}\Pi\left(\frac{x-n\Delta x}{\Delta x}\right)\Delta x.$



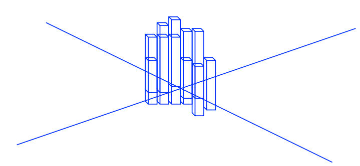
Representation of 2D Function

Similarly, we can write a 2D function as

$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(\xi,\eta)\delta(x-\xi,y-\eta)d\xi d\eta.$

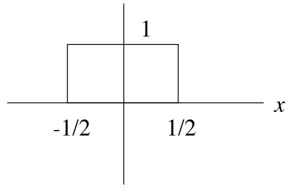
To gain intuition, consider the approximation

$g(x,y) \approx \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}g(n\Delta x, m\Delta y)\frac{1}{\Delta x}\Pi\left(\frac{x-n\Delta x}{\Delta x}\right)\frac{1}{\Delta y}\Pi\left(\frac{y-m\Delta y}{\Delta y}\right)\Delta x\Delta y.$



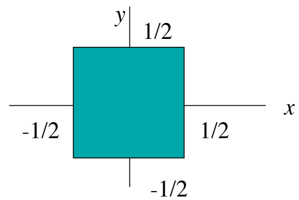
Rectangle Function

$$\Pi(x) = \begin{cases} 0 & |x| > 1/2 \\ 1 & |x| \le 1/2 \end{cases}$$



Also called rect(x)

$$\Pi(x,y) = \Pi(x)\Pi(y)$$



Computing Transforms

$$F(\delta(x)) = \int_{-\infty}^{\infty} \delta(x) e^{-j2\pi k_x x} dx = 1$$

$$F(\delta(x - x_0)) = \int_{-\infty}^{\infty} \delta(x - x_0) e^{-j2\pi k_x x} dx = e^{-j2\pi k_x x_0}$$

$$\begin{aligned} F\left(\Pi(x)\right) &= \int_{-1/2}^{1/2} e^{-j2\pi k_x x} dx \\ &= \frac{e^{-j\pi k_x} - e^{j\pi k_x}}{-j2\pi k_x} \\ &= \frac{\sin(\pi k_x)}{\pi k_x} = \mathrm{sin}c(k_x) \end{aligned}$$

Computing Transforms

$$F(1) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx = ???$$

Define $h(k_x) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx$ and see what it does under an integral.

$$\begin{aligned} \int_{-\infty}^{\infty} G(k_x) h(k_x) dk_x &= \int_{-\infty}^{\infty} G(k_x) \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx dk_x \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(k_x) e^{-j2\pi k_x x} dk_x dx \\ &= \int_{-\infty}^{\infty} g(-x) dx \\ &= G(0) \end{aligned}$$

Therefore, $F(1) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx = \delta(k_x)$

Basic Properties

Linearity

$$F\left[ag(x,y) + bh(x,y)\right] = aG(k_x,k_y) + bH(k_x,k_y)$$

Scaling

$$F\left[g(ax,by)\right] = \frac{1}{|ab|} G\left(\frac{k_x}{a}, \frac{k_x}{b}\right)$$

Shift

$$F\left[g(x-a, y-b)\right] = G(k_x,k_y) e^{-j2\pi(k_x a + k_y b)}$$

Modulation

$$F\left[g(x,y) e^{j2\pi(xa+yb)}\right] = G(k_x - a, k_y - b)$$

Linearity

The Fourier Transform is linear.

$$F\{ag(x)+bh(x)\}=aG(k_x)+bH(k_x)$$

$$F[ag(x,y)+bh(x,y)]=aG(k_x,k_y)+bH(k_x,k_y)$$

Computing Transforms

Similarly,

$$F\{e^{j2\pi k_0 x}\}=\delta(k_x-k_0)$$

$$F\{\cos 2\pi k_0 x\}=\frac{1}{2}\left(\delta(k_x-k_0)+\delta(k_x+k_0)\right)$$

$$F\{\sin 2\pi k_0 x\}=\frac{1}{2j}\left(\delta(k_x-k_0)-\delta(k_x+k_0)\right)$$

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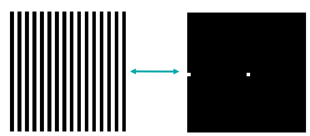
Examples

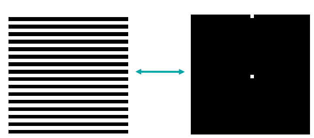
$$g(x,y)=1+e^{-j2\pi ax}$$

$$G(k_x,k_y)=\delta(k_x,k_y)+\delta(k_x+a)\delta(k_y)$$



$$g(x,y)=1+e^{j2\pi ay}$$

$$G(k_x,k_y)=\delta(k_x,k_y)+\delta(k_x)\delta(k_y-a)$$



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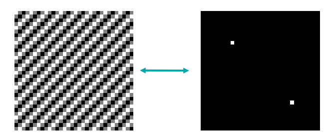
Examples



$$g(x,y)=\cos(2\pi(ax+by))$$

$$G(k_x,k_y)=\frac{1}{2}\delta(k_x-a)\delta(k_y-b)+\frac{1}{2}\delta(k_x+a)\delta(k_y+b)$$

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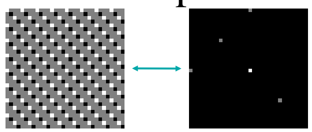
Examples

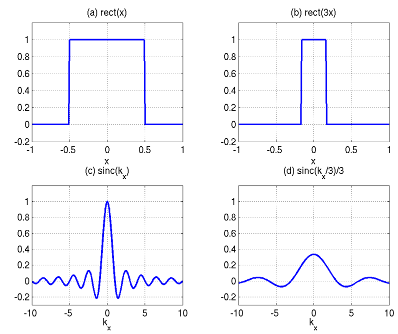


$$G(k_x,k_y) = \delta(k_x,k_y) + \delta(k_x+c)\delta(k_y) + \delta(k_x)\delta(k_y-d) + \frac{1}{2}\delta(k_x-a)\delta(k_y-b) + \frac{1}{2}\delta(k_x+a)\delta(k_y+b)$$

$g(x,y) = ???$

Scaling Theorem

$$F\{g(ax)\} = \frac{1}{|a|}G\left(\frac{k_x}{a}\right) \qquad F[g(ax,by)] = \frac{1}{|ab|}G\left(\frac{k_x}{a},\frac{k_x}{b}\right)$$



Separable Functions

$g(x,y)$ is said to be a separable function if it can be written as $g(x,y) = g_X(x)g_Y(y)$

The Fourier Transform is then separable as well.

$$G(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi(k_x x + k_y y)}dxdy$$
$$= \int_{-\infty}^{\infty} g_X(x)e^{-j2\pi k_x x}dx \int_{-\infty}^{\infty} g_Y(y)e^{-j2\pi k_y y}dy$$
$$= G_X(k_x)G_Y(k_y)$$

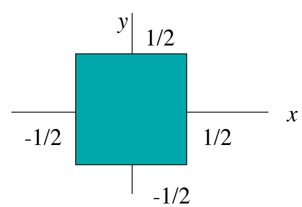
Example

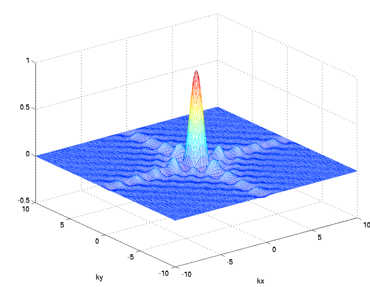
$g(x,y) = \Pi(x)\Pi(y)$

$G(k_x,k_y) = \mathrm{sinc}(k_x)\mathrm{sinc}(k_y)$

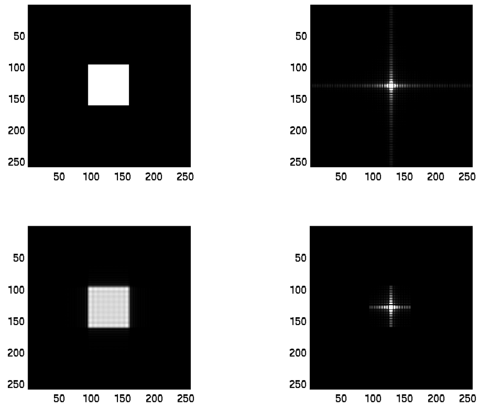
Example (sinc/rect)

Example
$g(x,y) = \Pi(x)\Pi(y)$
$G(k_x,k_y) = \mathrm{sinc}(k_x)\mathrm{sinc}(k_y)$





Example (sinc/rect)



Examples

$$g(x,y) = \delta(x,y) = \delta(x)\delta(y)$$
$$G(k_x,k_y) = 1$$

$$g(x,y) = \delta(x)$$
$$G(k_x,k_y) = \delta(k_y) \text{ !!!}$$

Duality

Note the similarity between these two transforms

$$F\left\{e^{j2\pi ax}\right\} = \delta(k_x - a)$$
$$F\left\{\delta(x-a)\right\} = e^{-j2\pi k_x a}$$

These are specific cases of duality

$$F\left\{G(x)\right\} = g(-k_x)$$

Application of Duality

$$F\left\{\text{sinc}(x)\right\} = \int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x} e^{-j2\pi k_x x} dx = ??$$

Recall that $F\left\{\Pi(x)\right\} = \text{sinc}(k_x)$.

Therefore from duality, $F\left\{\text{sinc}(x)\right\} = \Pi(-k_x) = \Pi(k_x)$