Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2013
CT/Fourier Lecture 2



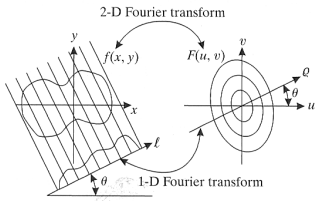
# Topics

- Projection Slice Theorem
- Fourier Transforms



# Projection Slice Theorem

$$
\begin{aligned}
G(\rho,\theta) &= \int_{-\infty}^{\infty} g(l,\theta) e^{-j2\pi\rho l}\, dl \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\delta(x\cos\theta + y\sin\theta - l)\, e^{-j2\pi\rho l}\, dx\, dy\, dl \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi\rho(x\cos\theta + y\sin\theta)}\, dx\, dy \\
&= F_{2D}\left[f(x,y)\right]\Big|_{u=\rho\cos\theta,\, v=\rho\sin\theta}
\end{aligned}
$$

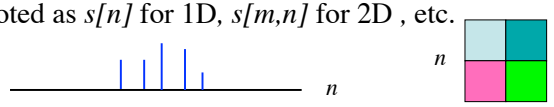



Prince&Links 2006

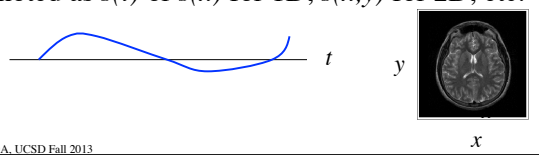


# Signals and Images

Discrete-time/space signal /image: continuous valued function with a discrete time/space index, denoted as *s[n]* for 1D, *s[m,n]* for 2D , etc.

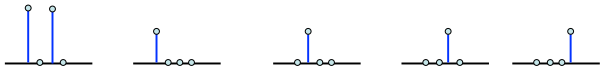



Continuous-time/space signal / image: continuous valued function with a continuous time/space index, denoted as *s(t)* or *s(x)* for 1D, *s(x,y)* for 2D, etc.

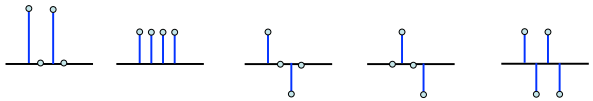


# 1D Signal Decomposition

$$\{2,0,2,0\} = 2\cdot\{1,0,0,0\} + 0\cdot\{0,1,0,0\} + 2\cdot\{0,0,1,0\} + 0\cdot\{0,0,0,1\}$$



$$\{2,0,2,0\} = a\cdot\{1,1,1,1\} + b\cdot\{1,0,-1,0\} + c\cdot\{0,1,0,-1\} + d\cdot\{1,-1,1,-1\}$$

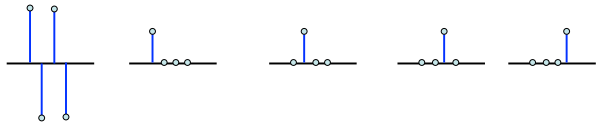


$$\{2,0,2,0\} = 1\cdot\{1,1,1,1\} + 0\cdot\{1,0,-1,0\} + 0\cdot\{0,1,0,-1\} + 1\cdot\{1,-1,1,-1\}$$

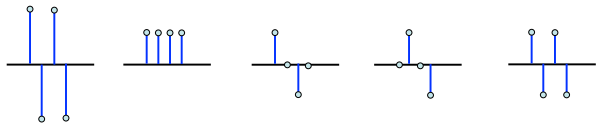


# 1D Signal Decomposition

$$\{2,-2,2,-2\} = 2\cdot\{1,0,0,0\} - 2\cdot\{0,1,0,0\} + 2\cdot\{0,0,1,0\} - 2\cdot\{0,0,0,1\}$$



$$\{2,-2,2,-2\} = 0\cdot\{1,1,1,1\} + 0\cdot\{1,0,-1,0\} + 0\cdot\{0,1,0,-1\} + 2\cdot\{1,-1,1,-1\}$$





# Eskimo Words for Snow

| | |
|---|---|
| tlapa | powder snow |
| tlacringit | snow that is crusted on the surface |
| kayi | drifting snow |
| tlapat | still snow |
| klin | remembered snow |
| naklin | forgotten snow |
| tlamo | snow that falls in large wet flakes |
| tlatim | snow that falls in small flakes |
| tlaslo | snow that falls slowly |
| tlapinti | snow that falls quickly |
| kripya | snow that has melted and refrozen |
| tliyel | snow that has been marked by wolves |
| tliyelin | snow that has been marked by Eskimos |
| | |
| tlalman | snow sold to German tourists |
| tlalam | snow sold to American tourists |
| tlanip | snow sold to Japanese tourists |
| tla-na-na | snow mixed with the sound of old rock and roll from a portable radio |
| depptla | a small snowball, preserved in Lucite, that had been handled by Johnny Depp |

http://www.mendosa.com/snow.html



# Image Compression

# 2D Image

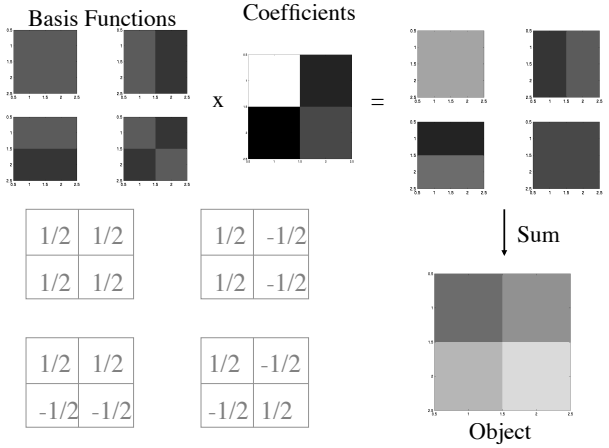
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$$



# Image Decomposition

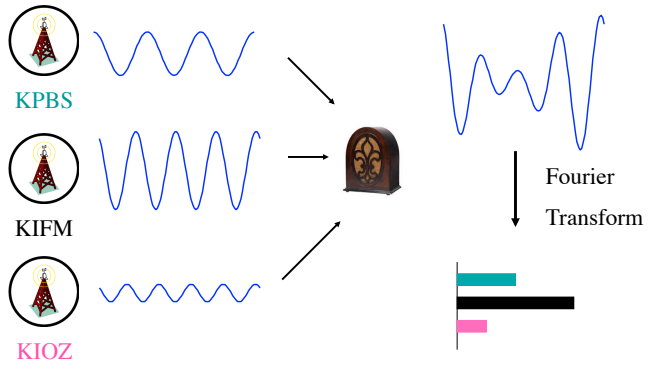
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$








# 1D Fourier Transform



# The Fourier Transform

Fourier Transform (FT)

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt = F\{g(t)\}$$

Inverse Fourier Transform

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df = F^{-1}\{G(f)\}$$



# Complex Numbers

$$j = \sqrt{-1}$$

$$j^2 = ?$$

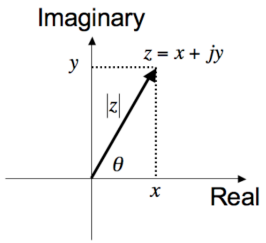
$$(3 + 2j)(3 - 2j) = ?$$

$$j^2 = -1$$

$$(3 + 2j)(3 - 2j) = 9 - 4j^2 = 13$$



# Complex Numbers




$$z = 2 + 1j$$

$$|z| = \sqrt{2^2 + 1} = \sqrt{5}$$

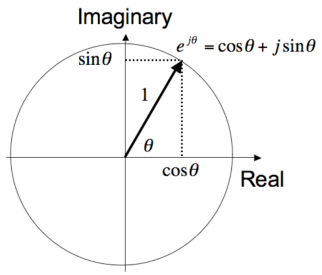
$$\theta = \tan^{-1}\left(\frac{1}{2}\right) = 30 \text{ degrees}$$



# Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$z = x + jy = |z| e^{j\theta}$$



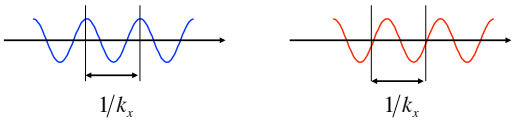
# 1D Fourier Transform

$$G(k_x) = \int_{-\infty}^{\infty} g(x)\exp\left(-j2\pi k_x x\right)dx$$

$$= \int_{-\infty}^{\infty} g(x)\cos(2\pi k_x x)dx - j\int_{-\infty}^{\infty} g(x)\sin(2\pi k_x x)dx$$

The part of $g(x)$ that "looks" like $\cos(2\pi k_x x)$

The part of $g(x)$ that "looks" like $\sin(2\pi k_x x)$






# Units

Temporal Coordinates, e.g. $t$ in seconds, $f$ in cycles/second

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt \qquad \text{Fourier Transform}$$

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df \qquad \text{Inverse Fourier Transform}$$

Spatial Coordinates, e.g. $x$ in cm, $k_x$ is spatial frequency in cycles/cm

$$G(k_x) = \int_{-\infty}^{\infty} g(x)e^{-j2\pi k_x x}dx \qquad \text{Fourier Transform}$$

$$g(x) = \int_{-\infty}^{\infty} G(k_x)e^{j2\pi k_x x}dk_x \qquad \text{Inverse Fourier Transform}$$



# 2D Fourier Transform

Fourier Transform

$$G(k_x,k_y) = F\left[g(x,y)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi\left(k_x x + k_y y\right)}dxdy$$

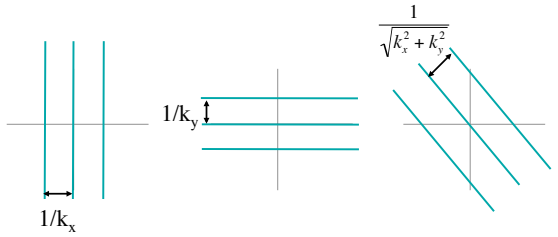
Inverse Fourier Transform

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x,k_y)e^{j2\pi\left(k_x x + k_y y\right)}dk_x dk_y$$



# Plane Waves

$$e^{j2\pi(k_x x + k_y y)} = \cos\left(2\pi(k_x x + k_y y)\right) + j\sin\left(2\pi(k_x x + k_y y)\right)$$




$\cos(2\pi k_x x)$ $\cos(2\pi k_y y)$ $\cos(2\pi k_x x + 2\pi k_y y)$

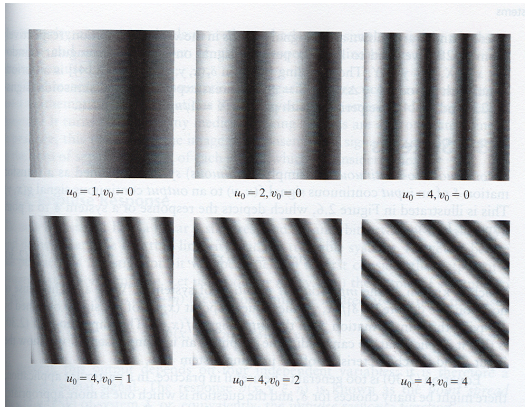
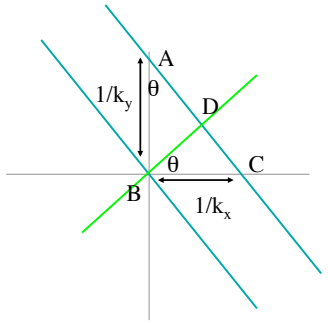



Figure 2.5 from Prince and Link



# Plane Waves




$$\Delta ABC \sim \Delta BDC$$

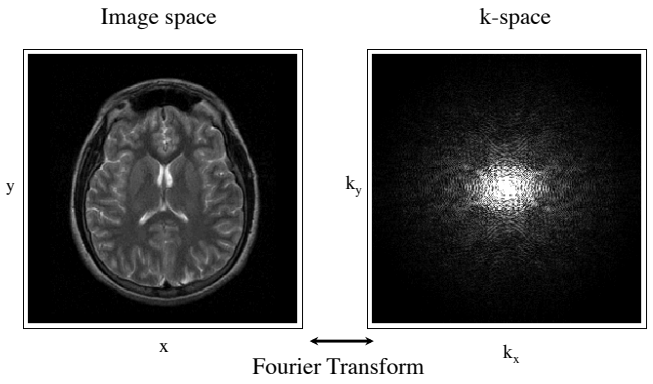
$$\frac{AC}{BC} = \frac{AB}{BD}$$

$$BD = AB\frac{BC}{AC} = \frac{\frac{1}{k_x}\frac{1}{k_y}}{\sqrt{\frac{1}{k_x^2}+\frac{1}{k_y^2}}} = \frac{1}{\sqrt{k_x^2+k_y^2}}$$

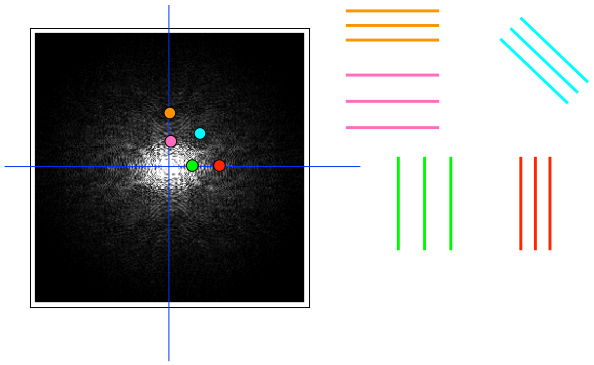
$$\theta = \arctan\left(\frac{k_y}{k_x}\right)$$

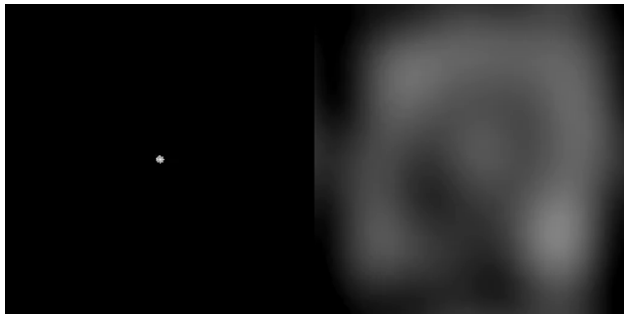


# k-space

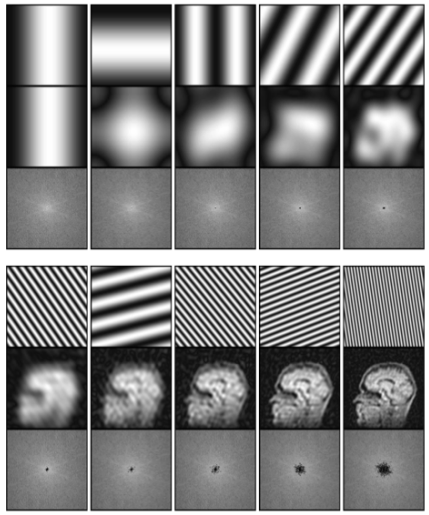






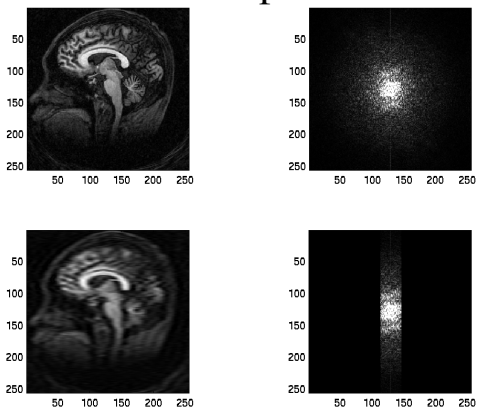




Hanson 2009

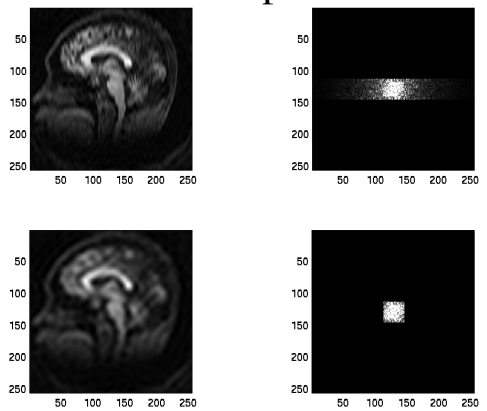


## Examples

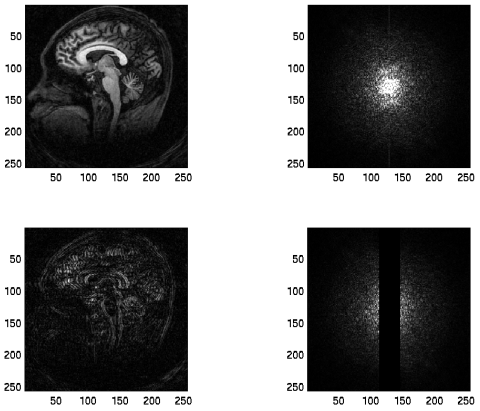





## Examples

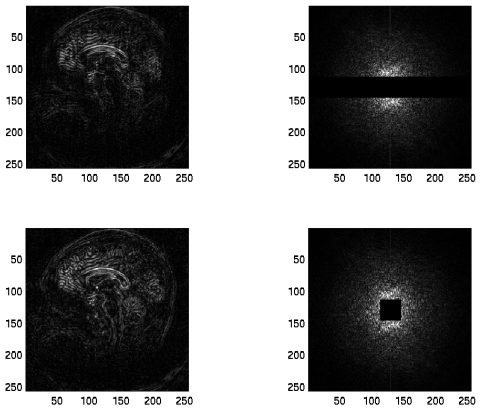


# Examples



TT Liu

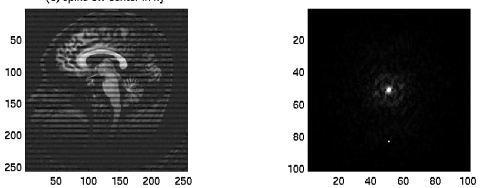
# Examples



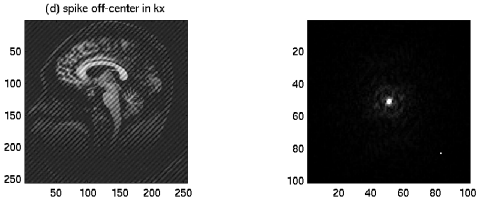
TT Liu, BE280A, UCSD Fall 2013

# Examples

(c) spike off-center in ky



(d) spike off-center in kx



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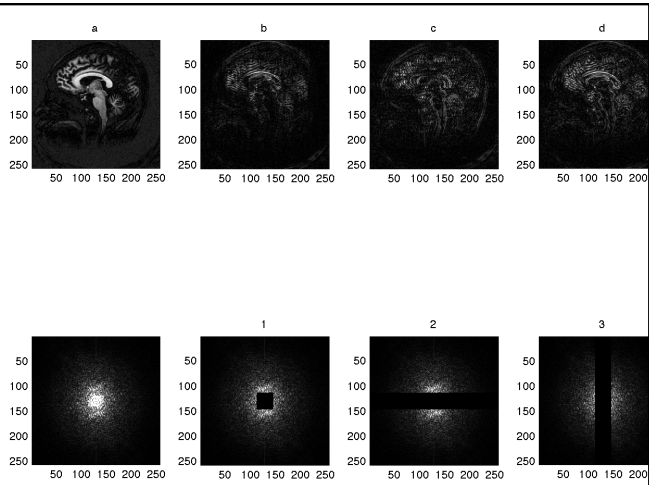


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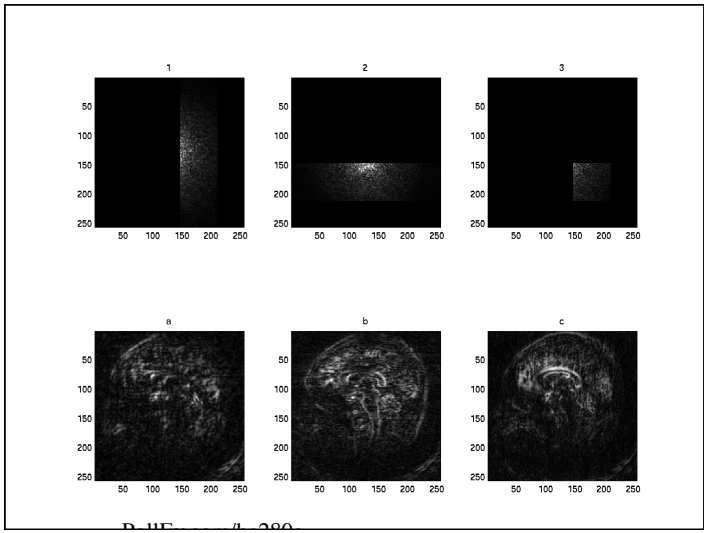


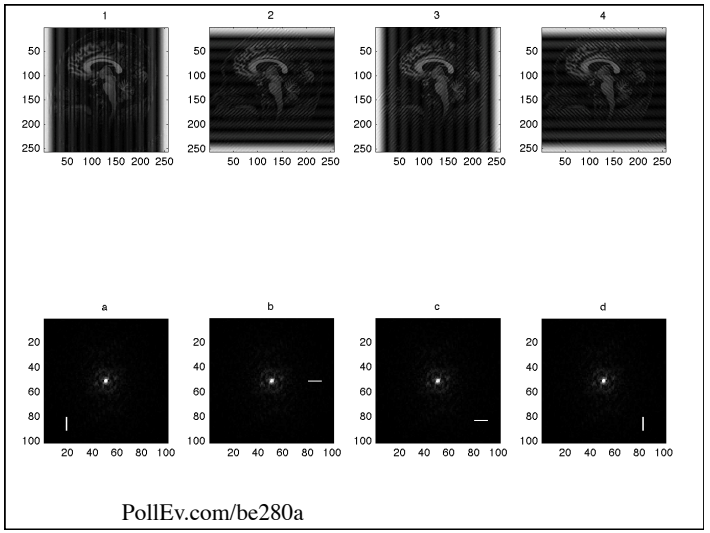


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