Bioengineering 280B
Comparative Biomedical Imaging

Spring Quarter 2005
Lecture 2
Linear Systems

Orthonormal Signal Expansions

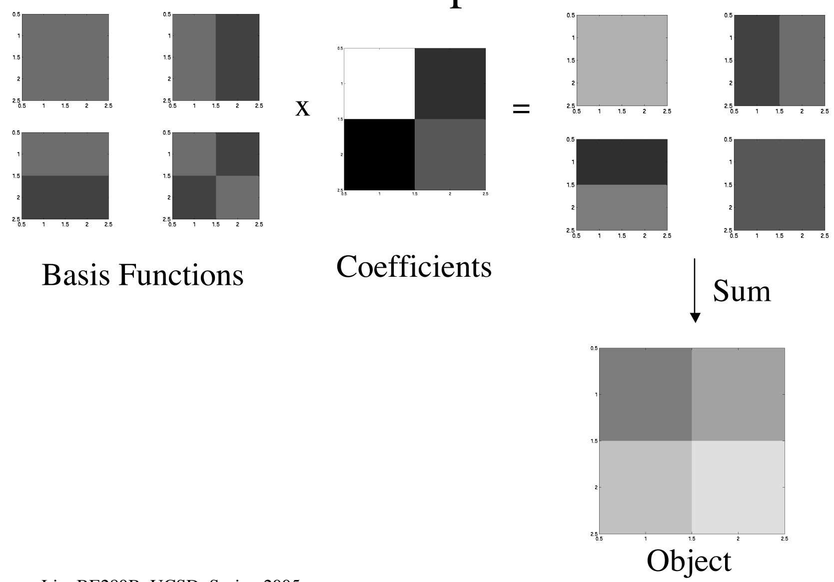
1D Discrete-Time Series Expansion

$$y[n] = \sum_{i=-\infty}^{\infty} c_i b_i[n] \qquad c_i = \langle b_i[n], y[n] \rangle$$

2D discrete expansion is

$$y[m,n] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_{k,l} b_{k,l}[m,n] \quad c_{k,l} = \langle b_{k,l}[m,n], y[m,n] \rangle$$

Example



Infinite Dimensional Orthonormal Expansions

Discrete-Time Series Expansion

$$y[n] = \sum_{i=-\infty}^{\infty} c_i b_i[n] \quad c_i = \langle b_i[n], y[n] \rangle$$

Continuous-Time Series Expansion

$$y(t) = \sum_{i=-\infty}^{\infty} c_i b_i(t) \quad c_i = \langle b_i(t), y(t) \rangle$$

Continuous-Time Integral Expansion

$$y(t) = \int_{-\infty}^{\infty} c_f b_f(t) df \quad c_f = \langle b_f(t), y(t) \rangle$$

Fourier Series Expansion

Basis functions are the complex exponentials

$$b_m(t) = \frac{1}{\sqrt{T}} e^{j2\pi m f_0 t} = \frac{1}{\sqrt{T}} \left(\cos 2\pi m f_0 t + j \sin 2\pi m f_0 t\right)$$

where f_0 is the fundamental frequency and $T_0 = 1/f_0$ is the fundamental period.

Are they orthonormal? Yes, over an interval defined by the period T_0.

$$\left\langle e^{j2\pi m f_0 t}, e^{j2\pi n f_0 t} \right\rangle = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} e^{j2\pi (m-n) f_0 t} dt = \delta[m-n]$$

Continuous - time series expansion is :

$$g(t) = \sum_{m=-\infty}^{\infty} c_m b_m(t) = \frac{1}{\sqrt{T}} \sum_{m=-\infty}^{\infty} c_m e^{j2\pi m f_0 t}$$

The basis coefficients are :

$$c_m = \left\langle \frac{1}{\sqrt{T}} e^{j2\pi m f_0 t}, g(t) \right\rangle = \frac{1}{\sqrt{T}} \int_{-T_0/2}^{T_0/2} g(t) e^{-j2\pi m f_0 t} dt$$

Fourier Series Expansion

Note that we can write the Fourier Series Expansion in a more familiar form as…

$$\begin{aligned} g(t) &= \frac{1}{\sqrt{T}} \sum_{m=-\infty}^{\infty} c_m e^{j2\pi m f_0 t} \\ &= \frac{1}{\sqrt{T}} \sum_{m=-\infty}^{\infty} c_m (\cos 2\pi m f_0 t + j \sin 2\pi m f_0 t) \\ &= \frac{1}{\sqrt{T}} \left[c_0 + \sum_{m=1}^{\infty} \left(c_m + c_{-m}\right) \cos 2\pi m f_0 t + j\left(c_m - c_{-m}\right) \sin 2\pi m f_0 t \right] \\ &= \frac{1}{\sqrt{T}} \left[c_0 + \sum_{m=1}^{\infty} a_m \cos 2\pi m f_0 t + b_m \sin 2\pi m f_0 t \right] \end{aligned}$$

The Fourier Transform

Basis functions are complex exponentials $b_f(t) = e^{j2\pi ft}$

Are they orthonormal?

$$\left\langle e^{j2\pi f_1 t}, e^{j2\pi f_2 t} \right\rangle = \int_{-\infty}^{\infty} e^{j2\pi (f_2 - f_1)t} dt = \delta(f_2 - f_1)$$

Continuous-time integral expansion is:

$$g(t) = \int_{-\infty}^{\infty} G(f) b_f(t) df = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df$$

The basis coefficients are:

$$G(f) = \left\langle e^{j2\pi ft}, g(t) \right\rangle = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt$$

The Fourier Transform

The Fourier Transform (FT) is simply given by the basis coefficients

$$G(f) = \left\langle e^{j2\pi ft}, g(t) \right\rangle = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt = F\{g(t)\}$$

The Inverse Fourier Transform is the continuous-time integral expansion for $g(t)$:

$$g(t) = \int_{-\infty}^{\infty} G(f) b_f(t) df = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df = F^{-1}\{G(f)\}$$

This can also be written as an inner product in Fourier Space

$$g(t) = \left\langle e^{-j2\pi ft}, G(f) \right\rangle$$

Units

Temporal Coordinates, e.g. t in seconds, f in cycles/second

$$G(f) = \langle e^{j2\pi ft}, g(t) \rangle = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt \quad \text{Fourier Transform}$$

$$g(t) = \langle e^{-j2\pi ft}, G(f) \rangle = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df \quad \text{Inverse Fourier Transform}$$

Spatial Coordinates, e.g. x in cm, k_x is spatial frequency in cycles/cm

$$G(k_x) = \langle e^{j2\pi k_x x}, g(x) \rangle = \int_{-\infty}^{\infty} g(x) e^{-j2\pi k_x x} dx \quad \text{Fourier Transform}$$

$$g(x) = \langle e^{-j2\pi k_x x}, G(k_x) \rangle = \int_{-\infty}^{\infty} G(k_x) e^{j2\pi k_x x} dk_x \quad \text{Inverse Fourier Transform}$$

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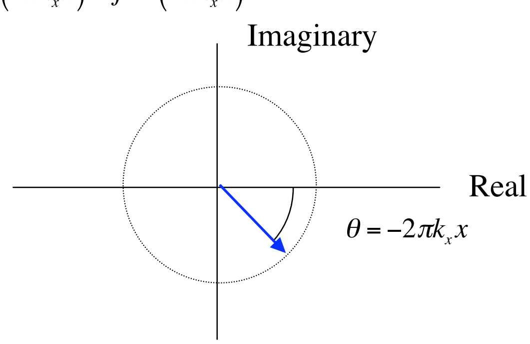
Phasor Diagram

Recall that a complex number has the form

$$z = a + jb = |c| \exp(j\theta) = |c| (\cos\theta + j\sin\theta)$$

where $|z| = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$

$$e^{-j2\pi k_x x} = \cos(2\pi k_x x) - j\sin(2\pi k_x x)$$



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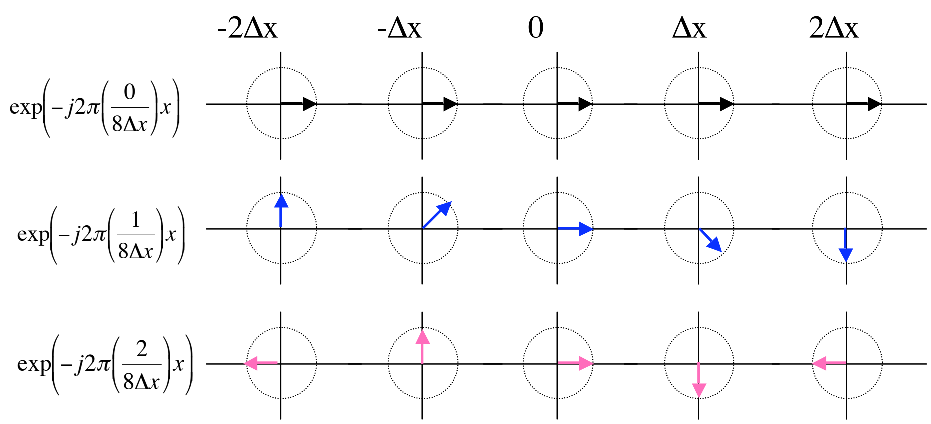
Interpretation



Computing Transforms

$$F(\delta(x)) = \int_{-\infty}^{\infty} \delta(x) e^{-j2\pi k_x x} dx = 1$$

$$F(\delta(x - x_0)) = \int_{-\infty}^{\infty} \delta(x - x_0) e^{-j2\pi k_x x} dx = e^{-j2\pi k_x x_0}$$

$$\begin{aligned} F\left(\Pi\left(x\right)\right) &= \int_{-1/2}^{1/2} e^{-j2\pi k_x x} dx \\ &= \frac{e^{-j\pi k_x} - e^{j\pi k_x}}{-j2\pi k_x} \\ &= \frac{\sin(\pi k_x)}{\pi k_x} = \operatorname{sinc}(k_x) \end{aligned}$$

Computing Transforms

$F(1) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx = ???$

Define $h(k_x) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx$ and see what it does under an integral.

$$\begin{aligned}
\int_{-\infty}^{\infty} G(k_x) h(k_x) dk_x &= \int_{-\infty}^{\infty} G(k_x) \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx dk_x \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(k_x) e^{-j2\pi k_x x} dk_x dx \\
&= \int_{-\infty}^{\infty} g(-x) dx \\
&= G(0)
\end{aligned}$$

Therefore, $F(1) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} dx = \delta(k_x)$

Computing Transforms

Similarly,

$$F\left\{e^{j2\pi k_0 x}\right\} = \delta(k_x - k_0)$$

$$F\left\{\cos 2\pi k_0 x\right\} = \frac{1}{2}\left(\delta(k_x - k_0) + \delta(k_x + k_0)\right)$$

$$F\left\{\sin 2\pi k_0 x\right\} = \frac{1}{2j}\left(\delta(k_x - k_0) - \delta(k_x + k_0)\right)$$

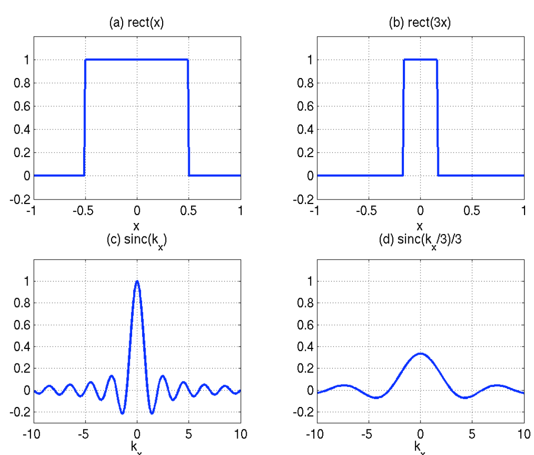
Linearity

The Fourier Transform is linear.

$$F\{ag(x)+bh(x)\} = aG(k_x)+bH(k_x)$$

Scaling Theorem

$$F\{g(ax)\} = \frac{1}{|a|}G\left(\frac{k_x}{a}\right)$$



Convolution/Modulation Theorem

$$F\{g(x)*h(x)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g(u)*h(x-u)du\right]e^{-j2\pi k_x x}dx$$
$$= \int_{-\infty}^{\infty} g(u)\int_{-\infty}^{\infty} h(x-u)e^{-j2\pi k_x x}dxdu$$
$$= \int_{-\infty}^{\infty} g(u)H(k_x)e^{-j2\pi k_x u}du$$
$$= G(k_x)H(k_x)$$

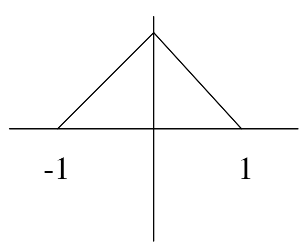
Convolution in the spatial domain transforms into multiplication in the frequency domain. Dual is modulation

$$F\{g(x)h(x)\} = G(k_x)*H(k_x)$$

Application of Convolution Thm.

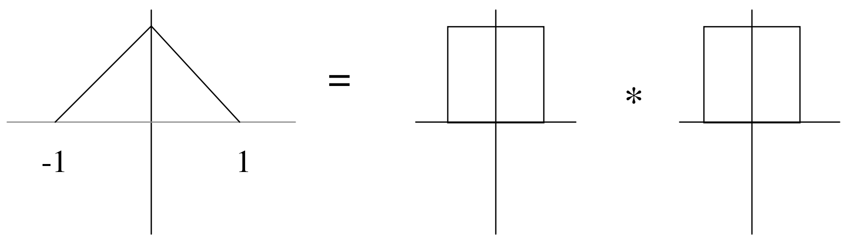
$$\Lambda(x) = \begin{cases} 1-|x| & |x|<1 \\ 0 & otherwise \end{cases}$$

$$F(\Lambda(x)) = \int_{-1}^{1}(1-|x|)e^{-j2\pi k_x x}dx = ??$$



Application of Convolution Thm.

$$\Lambda(x) = \Pi(x) * \Pi(x)$$
$$F(\Lambda(x)) = \mathrm{sinc}^2(k_x)$$



Modulation

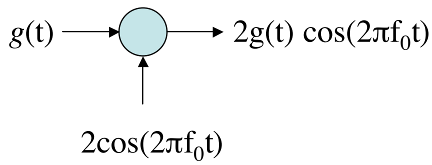
$$F\left[g(x)e^{j2\pi k_0 x}\right] = G(k_x) * \delta(k_x - k_0) = G(k_x - k_0)$$

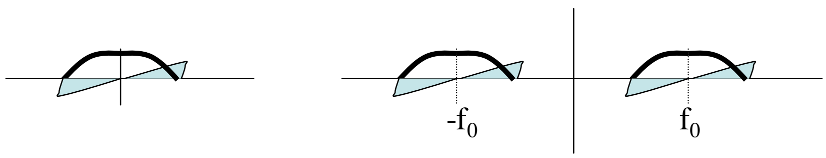
$$F\left[g(x)\cos(2\pi k_0 x)\right] = \frac{1}{2}G(k_x - k_0) + \frac{1}{2}G(k_x + k_0)$$

$$F\left[g(x)\sin(2\pi k_0 x)\right] = \frac{1}{2j}G(k_x - k_0) - \frac{1}{2j}G(k_x + k_0)$$

Example

Amplitude Modulation (e.g. AM Radio)





G(f) $\qquad$ G(f-f$_0$)+ G(f+f$_0$)

Parseval's Theorem

Recall that an orthonormal expansion preserves length or equivalently energy.

$$\int_{-\infty}^{\infty} |g(x)|^2 dx = \int_{-\infty}^{\infty} |G(k_x)|^2 dk_x$$

The more general form of this theorem is

$$\int_{-\infty}^{\infty} g(x)h^*(x)dx = \int_{-\infty}^{\infty} G(k_x)H^*(k_x)dk_x$$

Fourier Basis Functions

Recall that for 1D the basis functions are complex exponentials

$$b_{k_x}(x) = e^{j2\pi k_x x}$$

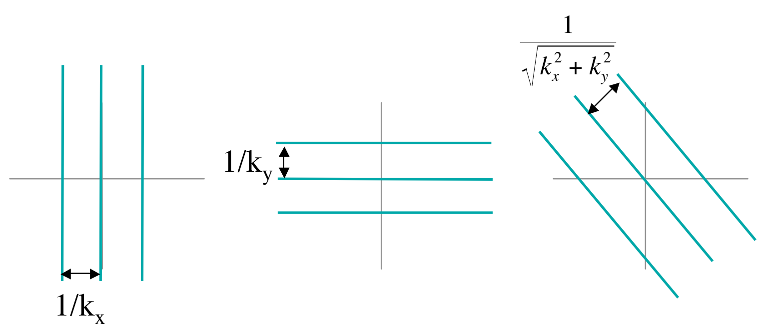
For 2D, we use the separable 2D functions

$$b_{k_x,k_y}(x,y) = b_{k_x}(x) b_{k_y}(y) = e^{j2\pi k_x x} e^{j2\pi k_y y} = e^{j2\pi(k_x x + k_y y)}$$

Are they orthonormal?

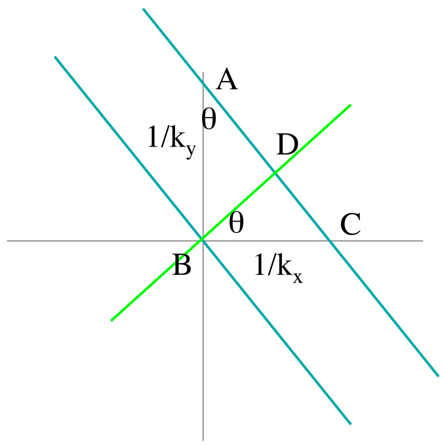
Plane Waves

$$e^{j2\pi(k_x x + k_y y)} = \cos\left(2\pi(k_x x + k_y y)\right) + j\sin\left(2\pi(k_x x + k_y y)\right)$$



$\cos(2\pi k_x x)$ $\cos(2\pi k_y y)$ $\cos(2\pi k_x x + 2\pi k_y y)$

Plane Waves



$$\Delta ABC \sim \Delta BDC$$

$$\frac{AC}{BC} = \frac{AB}{BD}$$

$$BD = AB\frac{BC}{AC} = \frac{\frac{1}{k_x}\frac{1}{k_y}}{\sqrt{\frac{1}{k_x^2}+\frac{1}{k_y^2}}} = \frac{1}{\sqrt{k_x^2+k_y^2}}$$

$$\theta = \arctan\left(\frac{k_y}{k_x}\right)$$

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2D Fourier Transform

Fourier Transform

$$G(k_x,k_y) = F\left[g(x,y)\right] = \left\langle e^{j2\pi(k_x x+k_y y)}, g\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi(k_x x+k_y y)}dxdy$$

Inverse Fourier Transform

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x,k_y)e^{j2\pi(k_x x+k_y y)}dk_x dk_y$$

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Separable Functions

$g(x,y)$ is said to be a separable function if it can be written as $g(x,y) = g_X(x)g_Y(y)$
The Fourier Transform is then separable as well.

$$\begin{aligned} G(k_x,k_y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi(k_x x + k_y y)}dxdy \\ &= \int_{-\infty}^{\infty} g_X(x)e^{-j2\pi k_x x}dx \int_{-\infty}^{\infty} g_Y(y)e^{-j2\pi k_y y}dy \\ &= G_X(k_x)G_Y(k_y) \end{aligned}$$

Example
$g(x,y) = \Pi(x)\Pi(y)$
$G(k_x,k_y) = \mathrm{sinc}(k_x)\mathrm{sinc}(k_y)$

Example (sinc/rect)

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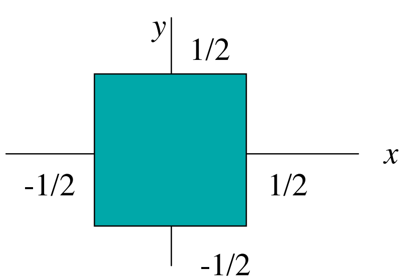
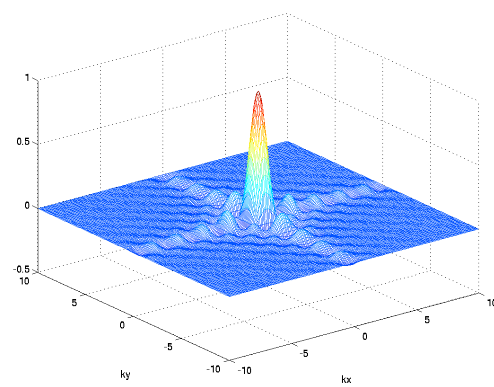
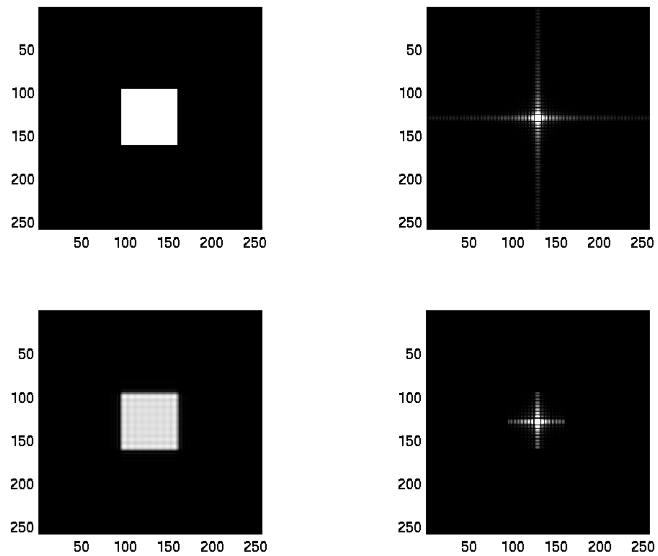
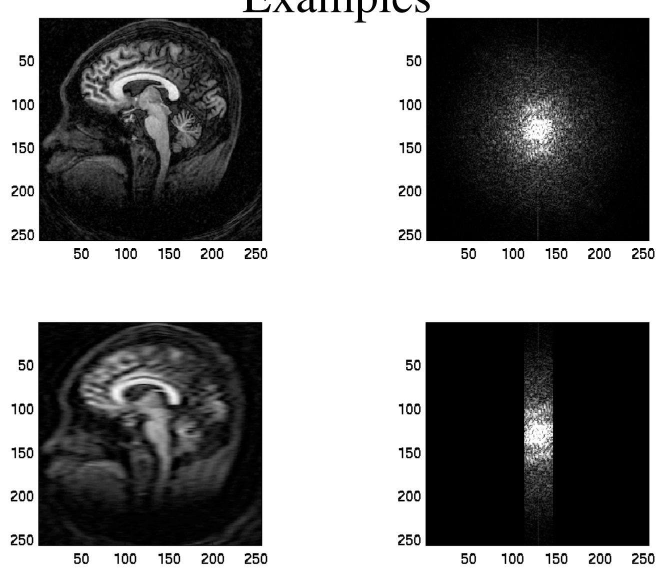




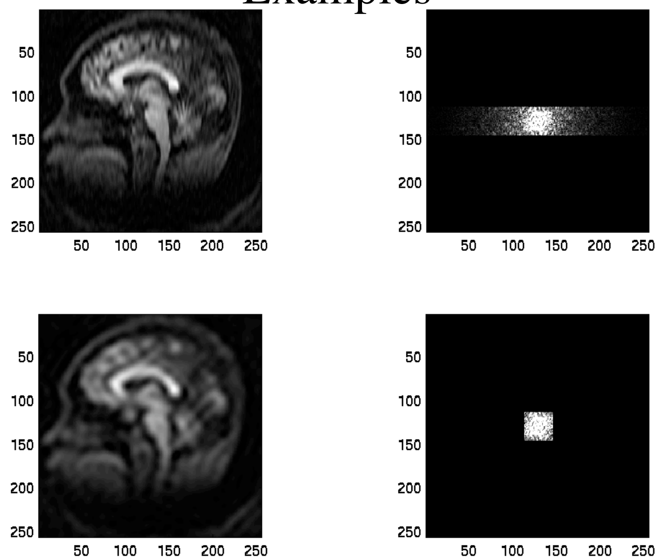
Example (sinc/rect)



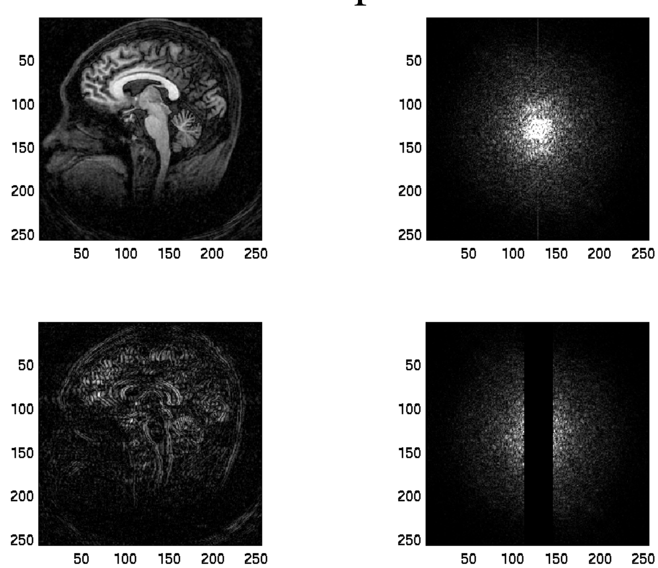
Examples



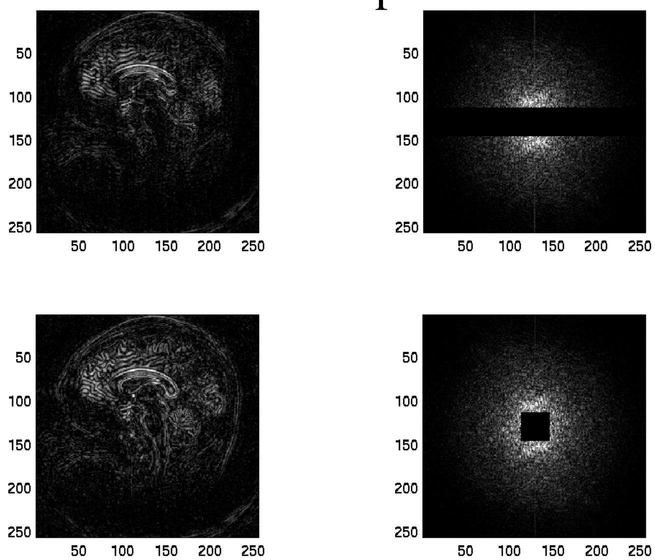
Examples



Examples



Examples



Examples

$$g(x,y) = \delta(x,y) = \delta(x)\delta(y)$$
$$G(k_x,k_y) = 1$$

$$g(x,y) = \delta(x)$$
$$G(k_x,k_y) = \delta(k_y) \text{ !!!}$$

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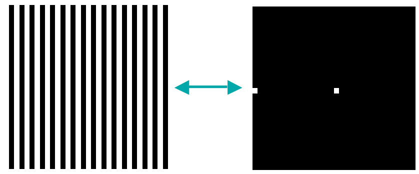
Examples

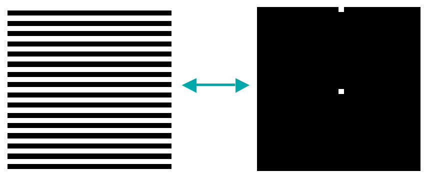
$$g(x,y) = 1 + e^{-j2\pi ax}$$

$$G(k_x,k_y) = \delta\left(k_x,k_y\right) + \delta(k_x + a)\delta(k_y)$$

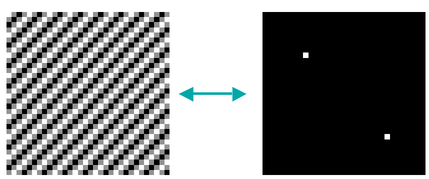


$$g(x,y) = 1 + e^{j2\pi ay}$$

$$G(k_x,k_y) = \delta(k_x,k_y) + \delta(k_x)\delta(k_y - a)$$



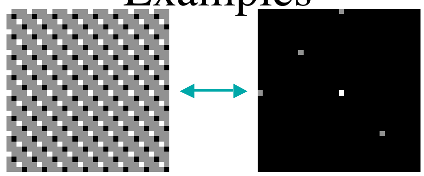
Examples



$$g(x,y) = \cos(2\pi(ax + by))$$

$$G(k_x,k_y) = \frac{1}{2}\delta(k_x - a)\delta(k_y - b) + \frac{1}{2}\delta(k_x + a)\delta(k_y + b)$$

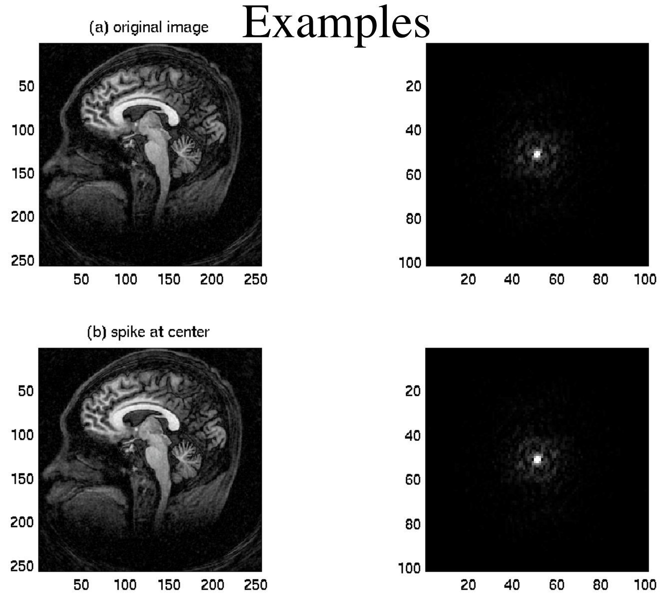
Examples



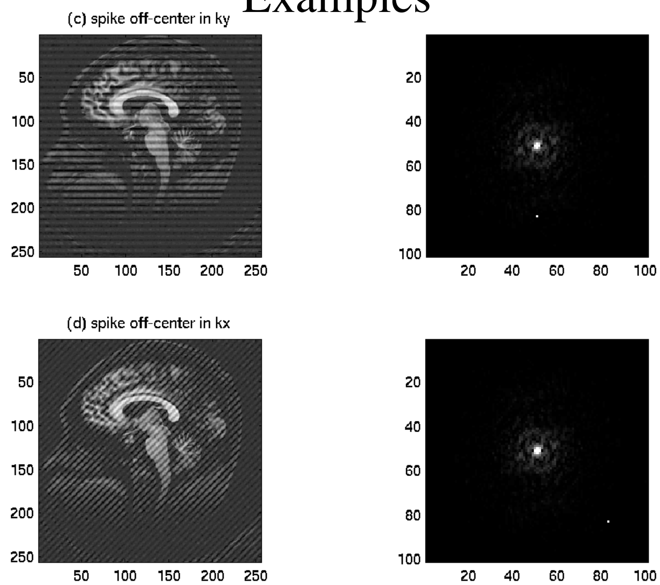
$$G(k_x,k_y) = \delta(k_x,k_y) + \delta(k_x + c)\delta(k_y) + \delta(k_x)\delta(k_y - d) + \frac{1}{2}\delta(k_x - a)\delta(k_y - b) + \frac{1}{2}\delta(k_x + a)\delta(k_y + b)$$

$$g(x,y) = ???$$

Examples



Examples



Basic Properties

Linearity

$$F\left[ag(x,y)+bh(x,y)\right]=aG(k_x,k_y)+bH(k_x,k_y)$$

Scaling

$$F\left[g(ax,by)\right]=\frac{1}{|ab|}G\left(\frac{k_x}{a},\frac{k_x}{b}\right)$$

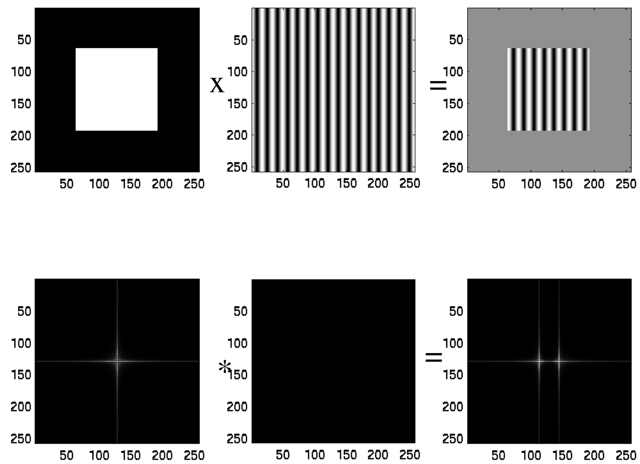
Shift

$$F\left[g(x-a,y-b)\right]=G(k_x,k_y)e^{-j2\pi(k_x a+k_y b)}$$

Modulation

$$F\left[g(x,y)e^{j2\pi(xa+yb)}\right]=G(k_x-a,k_y-b)$$

Modulation Example



Convolution/Multiplication

Convolution

$$F\left[g(x,y)**h(x,y)\right]=G(k_x,k_y)H(k_x,k_y)$$

Multiplication

$$F\left[g(x,y)h(x,y)\right]=G(k_x,k_y)**H(k_x,k_y)$$

Multiplication in one domain translates into convolution in the other domain.

Convolution Example